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THE ASYMPTOTIC BEHAVIOUR OF INTEGRAL FUNCTIONS

By J. CLUNIE (*Keele, North Staffs.*)

[Received 18 March 1954]

1. A GREAT deal of work has been done on the asymptotic values and paths of integral functions, but no one seems to have considered the following question explicitly. *If an integral function $f(z)$ possesses an asymptotic path L and a corresponding asymptotic value a, how quickly does $f(z) \to a$ as $z \to \infty$ along L?* In this paper I shall answer this question to the extent of giving a lower bound for the rate at which $f(z) \to a$.

Let $f(z)$ be an integral function having n asymptotic paths $L_1,..., L_n$ and n corresponding asymptotic values $a_1,..., a_n$, not necessarily all different. Any two L_i and L_j will divide the plane into two regions, and it is assumed that each of the regions contains a sequence of points $z_n \to \infty$ such that $f(z_n) \to \infty$. As a measure of how quickly $f(z) \to a_i$ along L_i I introduce the function $\epsilon(r)$ defined by

$$\epsilon(r) = \max_{\substack{i \\ 1 \leqslant i \leqslant n}} \Big\{ \max_{\substack{|z| \geqslant r \\ z \in L_i}} |f(z)-a_i| \Big\}.$$

From the definition of $\epsilon(r)$ it follows immediately that $\epsilon(r) \to 0$ as $r \to \infty$.

Although, as pointed out above, no explicit investigation of the question of how quickly $f(z) \to a$ along L has been made, there are two results, one due to Hayman (**1**) and the other due to Valiron and Collingwood (**5**) which can be adapted to give a lower bound for $\epsilon(r)$.† If

$$m(r) = \min_{|z|=r} |f(z)|$$

and 0 is an asymptotic value, then $\epsilon(r) \geqslant m(r)$. Thus it follows from Theorem 5 of (**1**) that

$$\epsilon(r) \geqslant [M(r)]^{-A_0}$$

for some large r. Hence by Lemma 7 of (**1**) it follows that, for all large r,

$$\epsilon(r) \geqslant [M(A_1 r)]^{-A_0}.$$

Here A_0, A_1 are absolute constants. The result I give is stronger than

† I am indebted to the referee for pointing out to me how Hayman's result is related to my own.

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the latter of these two results, but does not include the former. The result of Valiron and Collingwood is equivalent to

$$\epsilon(r) \geqslant \frac{\{M(r)\}^2}{M(kr)}\{1+o(1)\},$$

where k is an absolute constant, and for the value of k they give this result is certainly included in mine.

THEOREM. *If $f(z)$ possesses one or more asymptotic values, then*

$$\epsilon(r) \geqslant \frac{\{M(r)\}^4}{\{M(re^{2\pi})\}^3}\{1+o(1)\}.$$

When $f(z)$ possesses at least two distinct asymptotic values, a and b, then $\epsilon(r)$ also satisfies

$$\epsilon(r) \geqslant \frac{|a-b|^3}{16M(r)M(re^{\pi})}\{1+o(1)\}.$$

By making assumptions about the growth of $f(z)$ we could express the above inequalities in terms of the order, or of a proximate order.

2. To prove the theorem two lemmas are required. The proofs of these will be found in the references given.

LEMMA 1 [(**3**)]. *A function $\phi(z)$ which is regular in the square $ABCD$ of centre z_0 and for which $|\phi(z)| \leqslant m_1, m_2, m_3, m_4$ on AB, BC, CD, DA respectively satisfies*

$$|\phi(z_0)| \leqslant (m_1 m_2 m_3 m_4)^{\frac{1}{4}}.$$

LEMMA 2 [(**2**)]. *The annulus (r, R) in the z-plane crossed by n non-intersecting, piece-wise analytic curves can be mapped conformally onto the annulus (r, R') in the w-plane with $R' \geqslant R$ so that the images of the curves are radii. In general the mapping function $w = \chi(z)$ will consist of n separate analytic functions.*

Consider at first only one asymptotic path. By Lemma 2 the annulus $(r, re^{2\pi})$ in the z-plane can be mapped onto the annulus (r, R') in the w-plane with $R' \geqslant re^{2\pi}$ so that the part of the considered path in $(r, re^{2\pi})$ becomes a radius of (r, R'). Take $\zeta = \log w$ and consider the strip $\log r \leqslant \operatorname{re}\zeta \leqslant \log R'$, $0 \leqslant \operatorname{im}\zeta \leqslant 2\pi$ in the ζ-plane, where the sides of the strip on $\operatorname{im}\zeta = 0$ and $\operatorname{im}\zeta = 2\pi$ correspond to the above radius of (r, R') in the w-plane. Each point of the strip on the line $\operatorname{re}\zeta = \log(re^{\pi})$ is the centre of a square lying in the strip and with one side on either $\operatorname{im}\zeta = 0$ or $\operatorname{im}\zeta = 2\pi$. Hence, by Lemma 1, for such points,

$$|F(\zeta)| \leqslant \{M(re^{2\pi})\}^{\frac{3}{4}}\{\epsilon(r)\}^{\frac{1}{4}}\{1+o(1)\},$$

where $F(\zeta)$ is the function in the strip corresponding to $f(z)$ in the z-plane. Now at one such point $|F(z)| \geqslant M(r)$, and so

$$\epsilon(r) \geqslant \frac{\{M(r)\}^4}{\{M(re^{2\pi})\}^3}\{1+o(1)\},$$

which is the first part of the theorem.

Suppose now that $f(z)$ has two distinct asymptotic values a and b, and let the annulus (r, re^{π}) be mapped by $w = \chi(z)$ onto the annulus (r, R') $(R' \geqslant re^{\pi})$ so that the sections of the two corresponding paths within (r, re^{π}) become radii in (r, R'). Taking $\zeta = \log w$ and proceeding as before, we find that there is a point in the strip $\log r \leqslant \operatorname{re}\zeta \leqslant \log R'$, $0 \leqslant \operatorname{im}\zeta \leqslant \pi$ such that, by Lemma 1, for the corresponding point z_0 in the z-plane we have

$$|f(z_0)-a| \leqslant [\epsilon(r)|a-b|M(r)M(re^{\pi})]^{\frac{1}{2}}\{1+o(1)\},$$

$$|f(z_0)-b| \leqslant [\epsilon(r)|a-b|M(r)M(re^{\pi})]^{\frac{1}{2}}\{1+o(1)\}.$$

Hence
$$|a-b| \leqslant 2[\epsilon(r)|a-b|M(r)M(re^{\pi})]^{\frac{1}{2}}\{1+o(1)\},$$

and the second part of the theorem follows.

We show as follows that the result given by Valiron and Collingwood is included in the above theorem. We have, since $\epsilon(r)$ is non-increasing,

$$\begin{aligned}\{\epsilon(r)\}^3 &\geqslant \epsilon(r)\epsilon(re^{2\pi})\epsilon(re^{4\pi}) \\ &\geqslant \frac{\{M(r)\}^4 M(re^{2\pi})M(re^{4\pi})}{\{M(re^{6\pi})\}^3}\{1+o(1)\} \\ &> \frac{\{M(r)\}^6}{\{M(re^{6\pi})\}^3}\{1+o(1)\};\end{aligned}$$

and
$$\frac{\{M(r)\}^2}{M(re^{6\pi})}\{1+o(1)\}$$

is a better lower bound for $\epsilon(r)$ than

$$\frac{\{M(r)\}^2}{M(kr)}\{1+o(1)\}$$

since
$$e^{6\pi} < k = \left(\frac{11}{2\sqrt{3}-1}\right)^{36}.$$

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NOTE ON THE STEADY FLOW OF A FLUID PAST A THIN AEROFOIL

By D. S. JONES (*Manchester*)

[Received 20 March 1954]

Introduction

THE methods used in the linearized theory of flow past thin aerofoils are well known; they involve the use of conformal transformations and Fourier series [see, for example, Durand (**1**)]. The following note describes a method which is applicable when the aerofoil boundary is given by means of a polynomial and which should also be applicable to problems where the method of conformal transformation fails.

The technique used has been described elsewhere (**2**) but takes a less simple form here because the boundary conditions are more complicated.

Mathematical formulation of the problem

We consider the two-dimensional steady flow of an incompressible fluid past a thin aerofoil at a small angle of incidence to the main stream. We make the usual approximation of replacing the boundary condition on the aerofoil boundary by one on a straight line occupying a mean position; this line we shall call the *chord* of the aerofoil.

Let X, Y be Cartesian coordinates. Let the chord be of length l and occupy the interval $0 \leqslant X \leqslant l$, $Y = 0$. Let the velocity of the main stream be inclined at a small angle α to the X-axis and let U_0 be its magnitude. Then, if $U_0 \cos\alpha + u$, $U_0 \sin\alpha + v$ are the component parts of the velocity of the fluid, u and v are obtained from the velocity potential ϕ by means of $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, where $x = X/l$, $y = Y/l$. The excess pressure over hydrostatic p is given by the linearized equation

$$p = p_0 - \rho U_0\, \partial\phi/\partial x,$$

where ρ is the density of the fluid and p_0 the excess pressure at $x = -\infty$.

The equation satisfied by ϕ is

$$\nabla^2\phi = 0, \tag{1}$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ and ϕ must be such that the velocity normal to the aerofoil boundary is zero. I shall state this condition as

(i) $\partial\phi/\partial y = A_n x^{n-1}$ $(n \geqslant 1)$ on $0 \leqslant x \leqslant 1$, $y = 0$.

By taking a linear combination of such terms we can represent any

velocity distribution on the given aerofoil as a polynomial. In the particular case when the aerofoil is a flat plate, we take $n = 1$ and $A_n = -U_0 \sin\alpha$.

The velocity potential must be continuous except, possibly, across $y = 0$, $x > 0$. Since $\partial\phi/\partial y$ is clearly even in y, it follows that ϕ and $\partial\phi/\partial x$ are odd. Hence $\phi = 0$ and $\partial\phi/\partial x = 0$ on $y = 0$, $x < 0$. We now have the following conditions on ϕ:

(ii) the normal velocity $\partial\phi/\partial y$ is continuous across $y = 0$;

(iii) the pressure is continuous across $y = 0$, $x > 1$, i.e. $\partial\phi/\partial x$ is continuous across $y = 0$, $x > 1$;

(iv) $\phi = O(1)$, $|\mathrm{grad}\,\phi| = O(1/r^2)$ as $r \to \infty$, where $x = r\cos\theta$ $y = r\sin\theta$;

(v) the Kutta–Joukowski condition at the trailing edge, i.e.

$$\partial\phi/\partial x = O(1), \qquad \partial\phi/\partial y = O(1)$$

as the point of observation tends to the trailing edge;

(vi) the uniqueness condition that $\phi = O(1)$ as $r \to 0$.

In view of (ii), and of the fact that $\phi = 0$ on $y = 0$, $x < 0$ and that ϕ satisfies (1), it follows that $\phi = O(r^{\frac{1}{2}})$, $|\mathrm{grad}\,\phi| = O(r^{-\frac{1}{2}})$ as $r \to 0$.

The solution

Let $\Phi(s,\theta) = \int\limits_0^\infty \phi(r,\theta)r^{s-1}\,dr$, where s is the complex variable $\sigma+i\tau$.

On account of conditions (iv) and (vi), $\Phi(s,\theta)$ exists and is analytic in the strip $-\frac{1}{2} < \sigma < 0$.

Multiply (1) by r^{s+1} and integrate with respect to r from 0 to ∞. We obtain

$$\frac{\partial^2}{\partial\theta^2}\Phi(s,\theta)+s^2\Phi(s,\theta) = 0.$$

The solution of this equation such that (ii) is satisfied is

$$\Phi(s,\theta) = f(s)\sin s(\theta-\pi), \tag{2}$$

where $f(s)$ is a function of s only.

Designate the half-planes $\sigma > -\frac{1}{2}$, $\sigma < 0$ as the positive and negative half-planes respectively and denote quantities which are analytic in the positive and negative half-planes by the suffixes P and N respectively. Then

$$\begin{aligned}\Phi(s,\theta) &= \int\limits_0^1 \phi r^{s-1}\,dr + \int\limits_1^\infty \phi r^{s-1}\,dr \\ &= \Phi_P(s,\theta)+\Phi_N(s,\theta).\end{aligned} \tag{3}$$

Now, condition (iii) gives

$$\int_1^\infty r^s\left\{\frac{\partial}{\partial r}\phi(r,0)-\frac{\partial}{\partial r}\phi(r,2\pi)\right\}dr = 0,$$

i.e.

$$s\{\Phi_N(s,0)-\Phi_N(s,2\pi)\} = \phi(1,2\pi)-\phi(1,0)$$
$$= -\kappa, \tag{4}$$

where κ is independent of s. The constant κ can be identified with the circulation divided by l.

The combination of (2), (3), and (4) gives

$$-2f(s)\sin s\pi = \Phi(s,0)-\Phi(s,2\pi)$$
$$= \Psi_P-\kappa/s, \tag{5}$$

where $\Psi_P = \Phi_P(s,0)-\Phi_P(s,2\pi)$.

Also

$$\frac{\partial}{\partial\theta}\Phi(s,0) = sf(s)\cos s\pi,$$

and so condition (i) gives

$$\Psi'_N+A_n/(s+n) = sf(s)\cos s\pi, \tag{6}$$

where $\Psi'_N = \dfrac{\partial}{\partial\theta}\Phi_N(s,0)$. Eliminating $f(s)$ from (5) and (6), we obtain

$$\Psi'_N+A_n/(s+n) = -\tfrac{1}{2}K(s)\{\Psi_P-\kappa/s\}, \tag{7}$$

where

$$K(s) = s\cot s\pi.$$

Now

$$K(s) = K_P(s)/K_N(s),$$

where

$$K_P(s) = \frac{s!}{(s-\frac{1}{2})!}, \qquad K_N(s) = \frac{(-\frac{1}{2}-s)!}{(-s)!}. \tag{8}$$

K_P, K_N are analytic in the positive and negative half-planes respectively. As $|s| \to \infty$ in the positive half-plane, $K_P(s) \sim s^{\frac{1}{2}}$ and, as $|s| \to \infty$ in the negative half-plane, $K_N(s) \sim s^{-\frac{1}{2}}$.

Rewrite (7) as

$$K_N(s)\Psi'_N(s)+A_n\{K_N(s)-K_N(-n)\}/(s+n)-\tfrac{1}{2}\kappa K_P(0)/s$$
$$= -\tfrac{1}{2}K_P(s)\Psi_P(s)+\tfrac{1}{2}\kappa\{K_P(s)-K_P(0)\}/s-A_n K_N(-n)/(s+n).$$

The left-hand side is analytic in the negative half-plane; the right-hand side is analytic in the positive half-plane. Both sides have a strip in common and hence must equal an integral function. Now, as $|s| \to \infty$ in the negative half-plane, $\Psi'_N = o(1)$ and hence the left-hand side is $o(s^{-\frac{1}{2}})$. As $|s| \to \infty$ in the positive half-plane, the right-hand side is $o(s^{\frac{1}{2}})$.

It follows from the extension of Liouville's theorem that the integral function must be a constant. This constant must be zero on account of the behaviour in the negative half-plane. Hence

$$K_N(s)\Psi'_N(s) = -A_n\{K_N(s)-K_N(-n)\}/(s+n)+\tfrac{1}{2}\kappa K_P(0)/s.$$

As $|s| \to \infty$ in the negative half-plane,

$$\Psi'_N \sim \{A_n\, K_N(-n)+\tfrac{1}{2}\kappa K_P(0)\}s^{-\frac{1}{2}}.$$

But condition (v) implies that $\Psi'_N = O(1/s)$ as $|s| \to \infty$ in the negative half-plane. Hence

$$\kappa = -2K_N(-n)A_n/K_P(0) = -2\pi^{\frac{1}{2}}A_n(n-\tfrac{1}{2})!/n!. \tag{9}$$

It now follows from (2) and (6) that

$$\Phi(s,\theta) = -\frac{nA_n\, K_N(-n)\sin s(\theta-\pi)}{s^2(s+n)K_N(s)\cos s\pi},$$

and hence that

$$\phi(r,\theta) = -\frac{nA_n\, K_N(-n)}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{r^{-s}\sin s(\theta-\pi)}{s^2(s+n)K_N(s)\cos s\pi}\,ds, \tag{10}$$

where $-\frac{1}{2} < c < 0$.

The solution is now complete.

The lift

Equation (10) gives

$$\frac{\partial}{\partial r}\phi(r,0) = -\frac{nA_n\, K_N(-n)}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} \frac{r^{-s-1}\,ds}{(s+n)K_P(s)}$$

$$= -\frac{(n-\frac{1}{2})!\,A_n\, r^{n-1}}{(n-1)!\,\pi^{\frac{1}{2}}}\int\limits_{r}^{1} t^{-n-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt \quad (0 < r < 1)$$

by use of the Faltung theorem for Mellin transforms. The pressure distribution of the aerofoil may now be found from

$$p = p_0-\rho U_0\frac{\partial}{\partial r}\phi(r,0).$$

The lift per unit span can be found by integration of the pressure distribution or, more easily, by observing that it is

$$\rho l U_0\{\phi(1,0)-\phi(1,2\pi)\} = -2\rho l U_0\,\pi^{\frac{1}{2}}A_n(n-\tfrac{1}{2})!/n!$$

from (4) and (9).

The formula for the lift can be written as

$$-2\rho l U_0 A_n \int_0^1 r^{n-\frac{1}{2}}(1-r)^{-\frac{1}{2}}\,dr = -2\rho l U_0 \int_0^1 v_n r^{\frac{1}{2}}(1-r)^{-\frac{1}{2}}\,dr, \tag{11}$$

where $v_n = A_n r^{n-1}$. Clearly, in the general case where $\partial\phi/\partial y = V$ on the chord, V being a polynomial, (11) gives the lift correctly when v_n is replaced by V. Since any integrable $\partial\phi/\partial y$ can be approximated in mean by a polynomial, it follows that (11) with $\partial\phi/\partial y$ for v_n gives the lift for any $\partial\phi/\partial y$ on the chord.

The moment may be dealt with similarly.

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A DESCRIPTIVE ANALYSIS OF CONTINUOUS FUNCTIONS OF TWO REAL VARIABLES

J. R. RAVETZ (*Durham*)

[Received 30 March 1954]

1. Introduction

THE 'Denjoy theorem' for arbitrary functions of a single real variable shows that certain configurations of the Dini derivate numbers of such functions can occur only on a set of Lebesgue measure zero (**2**). Analogues to these results have been obtained by Haslam-Jones (**3**) for measurable functions of two real variables, and by Roger (**5**) for closed sets in Euclidean n-space.

The configurations of the Dini derivate numbers of continuous functions of a single real variable are subject to additional restrictions. Using the elementary fact that, for any real K, the set $\{x; D^+f(x) \geqslant K\}$ is a G_δ set, it is easy to prove that the set $G_x = \{x; D^+f(x) \neq D^-f(x)\}$ is a set of the first category: that is, the union of an enumerable family of nowhere-dense sets of points. This result was first proved by W. H. Young, using other methods. [See Theorem 4 of (**6**).]

In extending this 'descriptive analysis' to continuous functions of two real variables, we obtain analogues of the above result, and in addition we exhibit regularities in the tangential behaviour of such functions which have no analogues either in the one-variable case or in the 'metric' analysis of Haslam-Jones.

There are two ways of extending the notion of Dini derivate number to functions of two real variables. The straightforward extension is the *linear derivate* defined as follows (for convenience I use $z = x+iy$).

Let
$$\delta^\mu(\rho, z_0) = \sup_{0<r<\rho} r^{-1}[f(z_0+re^{i\mu})-f(z_0)]$$
and let
$$\partial^\mu f(z_0) = \lim_{\rho\to 0} \delta^\mu(\rho, z_0).$$

I call $\partial^\mu f(z_0)$ the *upper linear derivate* of $f(z)$ at z_0 in the direction μ.

Haslam-Jones obtained his analogue to the Denjoy theorem with the *directed derivate*, defined as follows.

Let $S_\mu(\rho, \eta)$ denote the open region consisting of points $z = re^{i\theta}$ for which
$$0 < r < \rho, \qquad \mu-\eta < \theta < \mu+\eta.$$
We define
$$B^\mu f(z_0) = B^\mu(\rho, \eta, z_0) = \sup_{h \in S_\mu(\rho,\eta)} |h|^{-1}[f(z_0+h)-f(z_0)].$$
Since $B^\mu f(z_0)$ is a non-decreasing function of ρ and η, the following limits exist:
$$\Delta^\mu(\eta, z_0) = \lim_{\rho\to 0} B^\mu(\rho, \eta, z_0),$$
$$D^\mu f(z_0) = \lim_{\eta\to 0} \Delta^\mu(\eta, z_0),$$
and we call $D^\mu f(z_0)$ the *upper directed derivate* of $f(z)$ at the point z_0, in the direction μ.

The lower linear and directed derivates $\partial_\mu f(z_0)$ and $D_\mu f(z_0)$ are defined in the naturally analogous fashion. For any point z_0 and direction μ, it is clear that $\partial^\mu f(z_0) \leqslant D^\mu f(z_0)$ and $\partial_\mu f(z_0) \geqslant D_\mu f(z_0)$. Let
$$g(t) = f(z_0+te^{i\mu});$$
$g(t)$ is then a continuous function of a single real variable. With $z' = z_0+t'e^{i\mu}$, we have
$$\partial^\mu f(z') = D^+g(t'), \qquad -\partial_{\mu+\pi} f(z') = D^-g(t').$$

We say that $f(z)$ has a *tangential singularity* at a point z_0 if there exists a direction μ such that at least one of the following conditions hold:

(i) $D^\theta f(z_0)$ or $D_\theta f(z_0)$ is discontinuous at $\theta = \mu$;

(ii) $D^\mu f(z_0) > \partial^\mu f(z_0)$ or $D_\mu f(z_0) < \partial_\mu f(z_0)$;

(iii) $D^\mu f(z_0) = +\infty$ or $D_\mu f(z_0) = -\infty$.

In § 3 we first study the behaviour of $f(z)$ in the neighbourhood of points at which it has tangential singularities, and then, using certain properties of Borel sets, we extend these local properties to give an overall description of the tangential behaviour of $f(z)$. The main result (Theorem 1) gives a decomposition of the domain of definition of $f(z)$ into two open sets U and V whose union is everywhere dense on that domain. At every point of V, $f(z)$ is free of tangential singularities, while for each point z_0 of a set residual on U, there exists an arc of directions $\Omega(z_0)$, which includes a closed semicircle, such that, for $\theta \in \Omega(z_0)$,
$$D^\theta f(z_0) = -D_{\theta+\pi} f(z_0) = +\infty.$$
In § 4, I obtain analogues to the result concerning the set G_x defined above. Finally, in § 5, I give counter-examples to two conjectures arising from the results of Theorem 1.

2. Notation

We define the *upper circular derivate* of $f(z)$ as

$$\mathscr{D}^*f(z_0) = \lim_{\rho \to 0} \sup_{|w| < \rho} |w|^{-1}[f(z_0+w)-f(z_0)].$$

This is the same as the function $L_f(z)$ defined by Rademacher and Stepanoff (**4**) in their work on the differentiability of functions $f(z)$. The *lower circular derivate*, $\mathscr{D}_*f(z)$, is defined in the naturally analogous fashion.

LEMMA 1. *For a continuous function $f(z)$ and a point z_0,*

(i) *$D^\theta f(z_0)$ is an upper semi-continuous function of θ;*

(ii)
$$\mathscr{D}^*f(z_0) = \sup_{0 \leqslant \theta \leqslant 2\pi} D^\theta f(z_0),$$

and analogous properties hold for the lower derivates at z_0.

The proof of (i) is given by Haslam-Jones (**3**); the proof of (ii) is elementary.

In the following discussion we shall need to deal with sets of pairs (z, θ), where $z = x+iy$ is a complex number and θ is a direction. We obtain a geometrical model for such sets in the following way. We can assume without loss of generality that $x > 1$; and we can represent the pair (z, θ) by the point $(x \cos\theta, y, x \sin\theta)$ in Euclidean three-space (which we denote by E_3), and give the pair (z, θ) the topology induced by the mapping.

We adopt the typographical convention that plane sets will be denoted by upper-case Roman letters in ordinary type, sets of directions by upper-case Greek letters, and sets of pairs (z, θ) by sans-serif upper-case Roman letters, as E, F.

By $\mathsf{A}(z_0)$ I shall mean the circle of points (z_0, θ) $(0 \leqslant \theta \leqslant 2\pi)$. By $\Lambda . \mathsf{A}(z_0)$ I shall mean the arc in E_3 of points (z_0, θ) with $\theta \in \Lambda$. If for some z_0, a subset of $\mathsf{A}(z_0)$, say $\mathsf{B} . \mathsf{A}(z_0)$, consists of a single arc (open or closed), then by $|\mathsf{B} . \mathsf{A}(z_0)|$ we shall mean the angle subtended by the arc of directions Λ where $\Lambda . \mathsf{A}(z_0) = \mathsf{B} . \mathsf{A}(z_0)$. The *z-projection* of a set B, written $(\mathsf{B})_z$, is defined to be the set of points z_0 such that $\mathsf{B} . \mathsf{A}(z_0)$ is not empty.

Since the set of directions is not an ordered set, I adopt the following notation. Given an arc of directions θ, which we denote by Θ, then we denote by $\mathscr{L}\Theta$ the 'lower' end-point, that end-point from which Θ is generated by a counter-clockwise rotation, and by $\mathscr{U}\Theta$ the other, 'upper' end-point. Given three directions λ, μ, ν, denote by Θ^* the open arc such that $\mathscr{L}\Theta^* = \lambda$, $\mathscr{U}\Theta^* = \mu$. Then by '$\lambda > \nu > \mu$' we shall mean that ν is included in Θ^*.

I now give an account of those aspects of the descriptive theory of sets of points which come into the subsequent arguments. All the properties discussed hold in spaces more general than those under consideration in the present work. I shall state all definitions of 'relative' concepts (such as density, first category, etc.) with open sets as the 'base' sets; this will be sufficient for our purposes.

Given an open set B and another set C, I say that C is *everywhere dense* on B if the closure of $B.C$ includes B. A set C is said to be *nowhere dense* on B if there is an open set H such that $H.C$ is empty and H is everywhere dense on B. A set C is said to be *of the first category* on B if $C = \bigcup_{i=1}^{\infty} C_i$, where each of the sets C_i is nowhere dense on B. If C is not of the first category on B, I say that C is *of the second category* on B.

There exist sets which are of the first category on themselves; the set of rational points on the unit interval is an example of this. It is a well-known result that open sets are of the second category on themselves [Carathéodory (**1**), § 79, Satz 7]. A set C is said to be *residual* on B if $B-C$ is a set of the first category on B. Clearly, a residual set is also a set of the second category. A subset C of B is said to be *closed with respect to* B if it contains all those of its points of accumulation which belong to B. Equivalent to this is that $C = B.F$, where F is a closed set.

A G_δ set is the intersection of a denumerable family of open sets; an F_σ set is the union of a denumerable family of closed sets. The complement of a G_δ set is an F_σ set, and vice versa. Open sets and closed sets are simultaneously G_δ and F_σ. By taking denumerable unions of G_δ sets and denumerable intersections of F_σ sets, we obtain sets denoted by $G_{\delta\sigma}$ and $F_{\sigma\delta}$ respectively. Thus, a set of the form $(G_\delta . F_\sigma)_\sigma$ is also of the form $(G_\delta . G_{\delta\sigma})_\sigma$, or simply $G_{\delta\sigma}$. The key properties of G_δ and F_σ sets are, roughly speaking, that an everywhere-dense G_δ set is a residual set, and that an F_σ set of the second category includes an open set. (See Lemmas 2 (*d*), 2 (*e*) below.)

LEMMA 2. *Let B be an open set.*

(*a*) *If a set C is not nowhere-dense on B, then there exists an open set H such that $B.H$ is not empty and C is everywhere dense on $B.H$.*

(*b*) *If $C = \bigcup_{i=1}^{\infty} C_i$ is of the second category on B, then so is one of the sets C_i.*

(*c*) *If $C = \bigcap_{i=1}^{\infty} C_i$, and each of the sets C_i is residual on B, then so is C.*

(*d*) *Suppose that* $C = \bigcup_{i=1}^{\infty} C_i$, *that each of the sets* C_i *is closed with respect to* B, *and that* C *is of the second category on* B. *Then there exists an open set* H *such that* $H.B$ *is not empty, and, for some* i, $C_i \supset H.B$.

(*e*) *Suppose that* C *is a* G_δ *set. Then* C *is residual on* B *if it is everywhere dense on* B.

(*f*) *Suppose that* $C = \bigcup_{i=1}^{N} C_i$, *where each of the sets* C_i *is nowhere dense on* B. *Then* C *is nowhere dense on* B.

(*g*) *Suppose that* C *is residual on* B, *and* $C \subset D \subset B$. *Then* D *is also residual on* B.

Proofs. (*a*) This is proved by Carathéodory (**1**) [cf. the last part of Satz 6, § 79].

(*b*) We assume the result to be false and let $C_i = \bigcup_{j=1}^{\infty} C_{ij}$, where each of the sets C_{ij} is nowhere dense on B. We then have

$$C = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} C_{ij}.$$

Enumerating the sets C_{ij} in the standard diagonal fashion gives us the result that C is of the first category.

(*c*) This is equivalent to (*b*); we prove it by taking complements and applying (*b*).

(*d*) We apply (*b*) and get a set C_i of the second category on B. This set C_i cannot be nowhere dense on B; applying (*a*) and using the assumption that C_i is closed with respect to B gives the desired result.

(*e*) We let

$$C = \bigcap_{i=1}^{\infty} C_i,$$

where C_i is open, for all i. Each of the sets C_i is everywhere dense on B; and it follows from the definitions that $B - C_i$ is nowhere dense on B, for all i. Then

$$B - C = \bigcup_{i=1}^{\infty} (B - C_i)$$

is a set of the first category.

(*f*) Since C is a finite union of sets C_i, we have

$$\bar{C} = \bigcup_{i=1}^{N} \bar{C}_i.$$

Each of the sets $\bar{C}_i$ is nowhere dense on B. If C were not nowhere-dense on B, then by (*a*) we should have $\bar{C} \supset H.B$. We should then have

$$H.B \subset \bigcup_{i=1}^{N} \bar{C}_i,$$

in contradiction to B being open.

(g) The set $B-D$ is included in the set $B-C$, which is of the first category on B. Using the trivial fact that a subset of a set of the first category is also of the first category, we prove the result.

3. Tangential singularities

I say that $f(z)$ has an *arc of tangential unboundedness* at z_0 if there exists an open arc of directions $\Psi(z_0)$ such that, for every direction μ in $\Psi(z_0)$, for every positive K, and for every neighbourhood N of z_0, the inequalities
$$D^{\mu}f(z') > K, \qquad D_{\mu+\pi}f(z'') < -K$$
are satisfied by points z', z'' lying in N (one or both of the points z', z'' may be identical with z_0). If $|\Psi(z_0)| \geqslant \pi$, then we refer to a *major arc of tangential unboundedness.*

LEMMA 3. *If a continuous function $f(z)$ has a tangential singularity at z_0, then $f(z)$ has a major arc of tangential unboundedness at z_0.*

We may suppose without loss of generality that the tangential singularity is in the behaviour of the upper derivates. Also, we let $z_0 = 0$, $f(0) = 0$. Let K be an arbitrary large positive number, and ρ and ϵ arbitrary small positive numbers. We shall construct an arc of directions Φ such that $|\Phi| = \pi - 2\epsilon$, and such that, for any direction ϕ in Φ, the inequalities
$$D^{\phi}f(z') > K, \qquad D_{\phi+\pi}f(z'') < -K \tag{1}$$
are satisfied by points $z' = z'(\phi)$, $z'' = z''(\phi)$ lying in the region $|z| < \rho$. The existence of a major arc of tangential unboundedness follows readily from this.

The methods of proof for the three types of tangential singularity are essentially the same; I now adopt a uniform notation encompassing the three cases. I show that, whatever the type of tangential singularity, there exist two directions λ and ν, distinct but arbitrarily close, such that
$$D^{\nu}f(0) > \partial^{\lambda}f(0). \tag{2}$$
If $D^{\theta}f(0)$ has a discontinuity, we let ν be the direction at which the discontinuity occurs; by the upper semicontinuity of $D^{\theta}f(0)$ [Lemma 1(i)] there exist directions λ arbitrarily close to ν at which $D^{\lambda}f(0) < D^{\nu}f(0)$, and (2) holds *a fortiori*. If, at some particular direction, the two upper derivates of $f(0)$ are unequal, and $D^{\theta}f(0)$ is continuous at this direction, then we denote the direction by λ; we then have $\partial^{\lambda}f(0) < D^{\lambda}f(0)$, and the existence of directions ν arbitrarily close to λ satisfying (2) follows from the continuity of $D^{\theta}f(0)$ at $\theta = \lambda$. If the derivates are unequal at some direction and $D^{\theta}f(0)$ is discontinuous there, then we ignore the

inequality of the derivates and obtain (2) from the discontinuity as above. Finally, we consider the case $\mathscr{D}^*f(0) = +\infty$. It is possible that $D^\theta f(0) = +\infty$ for all θ in an arc Θ such that $|\Theta| \geq \pi$. Then the existence of a major arc of tangential unboundedness follows trivially: the points z', z'' are all located at O. If this is not the case, we observe that the set $\{\theta; D^\theta f(0) = +\infty\}$ is closed [Lemma 1(i)], and let ν be a lower end-point of a closed arc of this set, or any direction of a nowhere-dense portion of the set. Then there are directions λ arbitrarily close to ν with

$$D^\lambda f(0) < +\infty,$$

and (2) is satisfied *a fortiori*.

Without loss of generality, we may strengthen inequality (2) to

$$\partial^\lambda f(0) < 0 < 2 < D^\nu f(0) \tag{3}$$

and we may specify the location and relative orientation of the directions λ and ν by

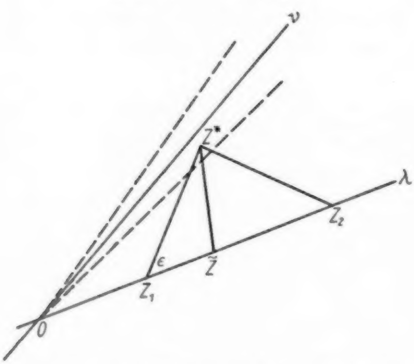
$$0 < \nu - 2\eta < \nu - \eta < \lambda < \nu < \nu + \eta < \tfrac{1}{2}\pi, \tag{4}$$

where η is a positive number satisfying

$$K \sin 2\eta < \sin \epsilon. \tag{5}$$

For convenience, we restrict K and ϵ by

$$K > 2, \qquad \epsilon < \tfrac{1}{4}\pi. \tag{6}$$



We now construct an isosceles triangle T, as in the figure, with the following properties:

(i) T lies in the region $x > 0$, $y > 0$, $|z| < \rho$;

(ii) the base of T lies on the ray $\arg z = \lambda$, and T has base angle ϵ;

(iii) $f(z^*) > |z^*|$, where z^* is the vertex of T; (7)

(iv) $f(\tilde{z}) < 0$, where $\tilde{z}$ is any point on the base of T. (8)

By condition (3), there exists a positive ρ' such that, for $0 < r < \rho'$,

$$f(re^{i\lambda}) < 0. \tag{9}$$

Let $\sigma = \frac{1}{4}\min(\rho, \rho')$, and let $\psi = \frac{1}{6}(\nu-\lambda)$. There exists a point z^* in $S_\nu(\sigma, \psi)$ (in the notation of § 1) such that $f(z^*) > |z^*|$; this follows from the definition of $D^\nu f(0)$ and (3). With this point z^* as vertex, we construct our triangle T by property (iii). Inequality (4), implying that

$$\epsilon > 2\eta > \nu-\lambda+\psi > \arg z^*-\lambda, \tag{10}$$

ensures that T lies in the first quadrant. Hence, if we denote the end-points of the base of T by z_1 and z_2 (with $|z_2| > |z_1|$), it is clear that $|z_2| < 2|z^*| < \frac{1}{2}\min(\rho, \rho')$, and property (i) is proved. Property (iv) follows from (9) and the last inequality. Property (iii) was fulfilled at the outset.

For any point $\tilde{z}$ on the base of T, the inequality

$$f(z^*)-f(\tilde{z}) > K|z^*-\tilde{z}| \tag{11}$$

is satisfied. To prove this, we observe the chain of inequalities

$$\frac{f(z^*)-f(\tilde{z})}{|z^*-\tilde{z}|} > \frac{f(z^*)}{|z^*-\tilde{z}|} > \frac{|z^*|}{|z^*-\tilde{z}|} \geqslant \frac{|z^*|}{|z^*-z_1|},$$

which hold by (8), (7), and the construction of T, respectively. Hence it is sufficient to show that $|z^*| > K|z^*-z_1|$. This follows from elementary trigonometry applied to the triangle with vertices O, z_1, z^*, together with (5) and (10): we apply the law of sines to the sides (O, z^*) and (z^*, z_1).

We now establish the existence of points z' and z'' as described in the first paragraph of this proof. We denote by Φ the arc $\lambda+\epsilon < \theta < \lambda+\pi-\epsilon$, and observe that $|\Phi| = \pi-2\epsilon$. For any direction ϕ of Φ, there is a point $\tilde{z}$ on the base of T such that $\phi = \arg(z^*-\tilde{z})$. For such a direction ϕ, we let $|z^*-\tilde{z}| = l$, and consider $g(t) = f(\tilde{z}+te^{i\phi})$ $(0 \leqslant t \leqslant l)$, a continuous function of a single real variable. We have $g(0) = f(\tilde{z})$, $g(l) = f(z^*)$, and, by (11), $g(l)-g(0) > Kl$. This implies the existence of points t', t'' on the open interval $(0, l)$ such that

$$D^+g(t') > K, \qquad D^-g(t'') > K.$$

Setting

$$z'(\phi) = \tilde{z}+t'e^{i\phi}, \qquad z''(\phi) = \tilde{z}+t''e^{i\phi},$$

we readily obtain inequalities (1). Since z' and z'' lie in T and hence in $|z| < \rho$, the proof is complete.

Note. If the tangential singularity of $f(z)$ at z_0 is of the form

$$D^\mu f(z_0) > \partial^\mu f(z_0)$$

or $D^{\theta}f(z_0)$ is discontinuous at $\theta = \mu$, then it is clear from the constructions that one end-point of the major arc of tangential unboundedness is μ itself.

LEMMA 4. *For a continuous function $f(z)$ and any number K, the set of points*

$$\mathsf{C}^K = \{z, \theta; D^{\theta}f(z) \geqslant K\}$$

is a G_δ set.

Let

$$H_{m,n}(z, \theta) = H(z, \theta, r_m, \theta_n) = r_m^{-1}[f(z+r_m \exp i[\theta-\theta_n])-f(z)],$$

where $\{r_m\}$ are the rational points on $(0, 1)$ and $\theta_n = 2\pi r_n$. Clearly, $H_{m,n}(z, \theta)$ is continuous with respect to (z, θ) and for any real K and integer l the set

$$\{z, \theta; H_{m,n}(z, \theta) > K-l^{-1}\}$$

is open. Defining $B^{\theta}f(z)$ as in § 1, we have

$$B^{\theta}f(z) = B^{\theta}(\rho, \eta, z) = \sup H_{m,n}(z, \theta),$$

where the supremum is taken over all m and n such that $0 < r_m < \rho$, $|\theta_n-\theta| < \eta$. Then $B^{\theta}f(z)$ is a lower semicontinuous function of (z, θ) and the set $\{z, \theta; B^{\theta}f(z) > K-l^{-1}\}$ is also open. Using the definition of $D^{\theta}f(z)$, we have that the set $\{z, \theta; D^{\theta}f(z) \geqslant K\}$ equals

$$\bigcap_{j=1}^{\infty} \bigcap_{l=1}^{\infty} \bigcap_{k=1}^{\infty} \{z, \theta; B^{\theta}(k^{-1}, j^{-1}, z) > K-l^{-1}\},$$

and so is a G_δ set.

Note. By analogous arguments, we obtain the additional result that the set $\{z, \theta; \partial^{\theta}f(z) \geqslant K\}$ is also a G_δ set. The intersection of C^K with the plane $\theta = \mu$ is a G_δ set; its projection on the z-plane is clearly a plane G_δ set; this latter set is $\{z; D^{\mu}f(z) \geqslant K\}$. Similarly, the set $\{z; \partial^{\mu}f(z) \geqslant K\}$ is a G_δ set. The fact that the set $\{x; D^+g(x) \geqslant K\}$ is a linear G_δ set, for continuous $g(x)$, may be deduced from these results, or may be proved in a manner analogous to that of the lemma. From this it readily follows, using Lemmas 2 (b) and 2 (d), that the set G_x is a linear set of the first category.

An argument even simpler than the one of the lemma gives the result that the plane set of points $P^K = \{z; \mathscr{D}^*f(z) \geqslant K\}$ is a G_δ set. By Lemma 1(ii), $P^K = (\mathsf{C}^K)_z$; thus we find a G_δ set projecting onto a G_δ set, instead of onto a general analytic set. This special situation is of importance in the main theorem.

Finally, we observe that by obvious modifications of the proof of the

lemma, we can obtain the result that the set $\{z, \theta; D_\theta f(z) \leqslant K\}$ is a G_δ set. Analogous results with lower derivates may be obtained for all the sets mentioned in this note.

LEMMA 5. *Let* G *be a* G_δ *set residual on an open set* K *in* E_3. *Let* L *denote the set of points* z_0 *such that* $\mathsf{G}.\mathsf{A}(z_0).\mathsf{K}$ *is not everywhere dense on* $\mathsf{A}(z_0).\mathsf{K}$. *Then the set* L *is of the first category.*

For each point z_0 of L there is an open set of directions $\Theta(z_0)$ such that
$$\Theta(z_0).\mathsf{A}(z_0) \subset \mathsf{K}.\mathsf{A}(z_0), \qquad \mathsf{G}.\Theta(z_0).\mathsf{A}(z_0) \text{ is empty.}$$

Let $\mathsf{K}-\mathsf{G} = \bigcup_{i=1}^{\infty} \mathsf{F}_i$, where each of the sets F_i is closed with respect to K. For each z_0 and i, the set $\mathsf{F}_i.\mathsf{A}(z_0)$ is closed with respect to $\mathsf{K}.\mathsf{A}(z_0)$. By the definition of $\Theta(z_0)$, for each $z_0 \in L$, we have
$$\Theta(z_0).\mathsf{A}(z_0) \subset \bigcup_{i=1}^{\infty} \mathsf{F}_i.\mathsf{A}(z_0).$$
Since $\Theta(z_0).\mathsf{A}(z_0)$ is open on $\mathsf{A}(z_0)$, it is a set of the second category on itself; hence not all the sets $\mathsf{F}_i.\Theta(z_0).\mathsf{A}(z_0)$ can be nowhere-dense on $\mathsf{A}(z_0)$. One of these sets is therefore everywhere dense on a sub-arc of $\Theta(z_0).\mathsf{A}(z_0)$; being closed with respect to $\mathsf{A}(z_0)$, it includes that sub-arc. Hence, for each point z_0 of L, there is an integer i and an arc $\Theta'(z_0)$ such that
$$\Theta'(z_0).\mathsf{A}(z_0) \subset \mathsf{F}_i.\mathsf{A}(z_0).$$

Denote by $\{\theta_j\}$ the set of rational directions, by $\{k\}$ the positive integers, and by Γ_{jk} the arc $|\theta-\theta_j| < 2\pi k^{-1}$. For each point z_0 of L, one of the arcs $\Gamma_{jk} \subset \Theta'(z_0)$. Hence, denoting by L_{ijk} the set
$$\{z; z \in L, \Gamma_{jk}.\mathsf{A}(z) \subset \mathsf{F}_i.\mathsf{A}(z)\},$$
we have $L = \bigcup_{i,j,k=1}^{\infty} L_{ijk}$.

If L is a set of the second category, then so must be one of the sets L_{ijk} [Lemma 2 (*b*)]. Denote such a set L_{ijk} by L^*, and let Γ and F denote the arc Γ_{jk} and closed set F_i corresponding to it. The set L^* cannot be nowhere-dense on $(\mathsf{K})_z$ (an open set); hence there is an open set $D \subset (\mathsf{K})_z$ such that L^* is everywhere dense on D (Lemma 2 (*a*)). Let T denote the three-dimensional region of points $\{z, \theta; z \in D, \theta \in \Gamma\}$. From the construction, we have $\mathsf{F}.\mathsf{T}$ everywhere dense on T. Since F is closed on K, $\mathsf{T} \subset \mathsf{F} \subset \mathsf{K}-\mathsf{G}$. Thus, $\mathsf{K}-\mathsf{G}$ includes an open set and so is not a set of the first category on K. This is the desired contradiction; hence L is a set of the first category.

LEMMA 6. *Let* (i) J *be a set of arcs of the form* $\Lambda(z).\mathsf{A}(z)$, *for all points* z *of a set* E *everywhere dense on a plane region* R, *and where* $|\Lambda(z)| \geqslant \alpha > 0$, *all* $z \in E$; (ii) K *be the interior of the closure of* J; (iii) F *be the plane set of points* z_0 *such that* $\mathsf{K}.\mathsf{A}(z_0)$ *includes an open arc subtending an angle at least* α. *Then* F *is residual on* R.

If, in addition, the set E *is residual on the region* R, *there exists a subset* G *of* E *also residual on* R *such that at all points* z_0 *of* G, $\Lambda(z_0).\mathsf{A}(z_0)$ *is included in the closure of* $\mathsf{K}.\mathsf{A}(z_0)$.

Let Θ_{pq} denote the arc

$$2\pi p/q \leqslant \theta < 2\pi(p+1)/q \quad (0 \leqslant p < q;\ 1 < q).$$

Let E_{pq} denote the subset of E such that $\mathscr{L}\Lambda(z) \in \Theta_{pq}$ (see § 2 for notation). For every $q > 1$, we have $E = \bigcup_{p=0}^{q-1} E_{pq}$. Let E^*_{pq} denote the interior of the closure of E_{pq}, and let $E^*_q = \bigcup_{p=0}^{q-1} E^*_{pq}$. For each $q > 1$, the open set E^*_q is everywhere dense on R. If this were not the case, then for some integer Q there would exist an open set R' such that $R'.E^*_Q$ is empty, and hence $R'.E^*_{pQ}$ $(0 \leqslant p < Q)$ is empty. Then, for each p, E_{pQ} would be nowhere dense on R', and $E.R'$, being the finite union of nowhere-dense sets, would also be nowhere dense on R' [Lemma 2(*f*)]. This contradicts the assumption that E is everywhere dense on R.

Let q_0 be an integer greater than $4\pi/\alpha$; we assume now that $q \geqslant q_0$. Denote by Φ_{pq} the arc

$$\mathscr{U}\Theta_{pq} < \theta < \mathscr{L}\Theta_{pq}+\alpha;$$

we have $|\Phi_{pq}| = \alpha - 2\pi/q$. For each point z_0 of E_{pq}, $\Phi_{pq} \subset \Lambda(z_0)$. Denote by E_{pq} and E^*_{pq} the sets

$$\{z, \theta;\ z \in E_{pq}, \theta \in \Phi_{pq}\}, \quad \{z, \theta;\ z \in E^*_{pq}, \theta \in \Phi_{pq}\}.$$

From the definitions, we have $\mathsf{E}_{pq} \subset \mathsf{J}$. Also, from the simple structure of E^*_{pq}, it is clear that E^*_{pq} equals the interior of the closure of E_{pq}. Hence E^*_{pq} is included in the interior of the closure of J; that is, $\mathsf{E}^*_{pq} \subset \mathsf{K}$.

Let z_0 be a point of some set E^*_{pq}. Then $\mathsf{K}.\mathsf{A}(z_0) \supset \mathsf{E}^*_{pq}.\mathsf{A}(z_0)$; this latter set is a single arc subtending an angle $\alpha - 2\pi/q$. Let $E^* = \bigcap_{q=q_0}^{\infty} E^*_q$; since the sets E^*_q are open sets everywhere dense on R, their complements are nowhere dense on R, and so the set E^* is residual on R. For any point z_0 of E^*, $\mathsf{K}.\mathsf{A}(z_0)$ includes arcs $\mathsf{E}^*_{pq}.\mathsf{A}(z_0)$ for all q; hence it includes an open arc subtending an angle at least α. Thus E^* is a subset of the

set F; since E^* is residual on R, then F is also [Lemma 2 (g)]; this is the desired result.

We now prove the second part of the lemma. For each point z_0 of R denote by $\Sigma(z_0)$ the open set of directions such that $\Sigma(z_0).\mathsf{A}(z_0)$ is the complement (with respect to $\mathsf{A}(z_0)$) of the closure of $\mathsf{K}.\mathsf{A}(z_0)$. Suppose the second part of the lemma to be false; we then assume E residual on R, and there exists a set T with the following properties: T is of the second category on R, and, at all points z_0 of T, $\Lambda(z_0).\Sigma(z_0)$ is not empty.

Let Γ_{jk} denote the arc $|\theta-\theta_j| < 2\pi/k$, where $\{\theta_j\}$ are the rational directions, and $\{k\}$ are the positive integers. Let T_{jk} denote the subset of T at whose points $\Lambda(z_0).\Sigma(z_0) \supset \Gamma_{jk}$. We have $T = \bigcup_{j,k=1}^{\infty} T_{jk}$; one of the sets T_{jk} must be of the second category on R [Lemma 2 (b)]. Denote it by T', and let Γ be the arc Γ_{jk} corresponding to it. There is a region R' on which T' is everywhere dense [Lemma 2 (a)]; denote by T the set $\{z,\theta; z \in T', \theta \in \Gamma\}$ and by T^* the set $\{z,\theta; z \in R', \theta \in \Gamma\}$. Since $\Gamma \subset \Lambda(z_0)$, for $z_0 \in T'$, it is clear that T is a subset of the original set J; it is also clear that T is everywhere dense on T^*. Hence $\mathsf{T}^* \subset \mathsf{K}$, and so, for all points z_0 of T', $\mathsf{T}^*.\Sigma(z_0).\mathsf{A}(z_0)$ is empty. Since, for $z_0 \in T'$,

$$\mathsf{T}^*.\mathsf{A}(z_0) = \Gamma.\mathsf{A}(z_0) \subset \Lambda(z_0).\Sigma(z_0).\mathsf{A}(z_0),$$

we have the desired contradiction, and so the second part of the lemma is proved.

THEOREM 1. *Let $f(z)$ be a continuous function defined on a plane region R. Then there are two open sets U and V whose union is everywhere dense on R, with the following properties: for each point z_0 of V, and for all directions θ, $D^\theta f(z_0)$ and $D_\theta f(z_0)$ are continuous, bounded, and equal to $\partial^\theta f(z_0)$ and $\partial_\theta f(z_0)$ respectively; for each point z_1 of a set residual on U, there is a closed arc $\Omega(z_1)$, including a semicircle, such that, for θ in $\Omega(z_1)$,*

$$D^\theta f(z_1) = -D_{\theta+\pi} f(z_1) = +\infty.$$

We let B be the set of points where $f(z)$ has a tangential singularity of any kind. Then we define $V = R-\bar{B}$, $U = R-\bar{V}$. It is clear that $(U+V)$ is everywhere dense on R and that B is everywhere dense on U. The stated properties for the set V follow directly from its definition: the boundedness of $D^\theta f(z_0)$ and $D_\theta f(z_0)$ results from their finiteness and continuity on the full circle of directions. We now prove the stated properties of the set U.

Let $\Psi(z_0)$ be the major arc of tangential unboundedness defined by

Lemma 3 at points z_0 of B. Let $\mathsf{C}^K = \{z, \theta; D^\theta f(z) \geqslant K\}$, and let $\mathsf{C}^* = \bigcap_{K=1}^{\infty} \mathsf{C}^K$. Let $\Omega(z_0)$ be the closure of $\Psi(z_0)$, for points z_0 of B. We shall show that there exists a set W residual on U at all of whose points z_1

$$\mathsf{C}^*.\mathsf{A}(z_1) \supset \Omega(z_1).\mathsf{A}(z_1).$$

This is the desired result for upper derivates.

We first show that B is residual on U. From the definitions, B is everywhere dense on U. Then, by Lemma 3, the sets

$$P^K = \{z; \mathscr{D}^* f(z) \geqslant K\}$$

are also everywhere dense on U. These sets are G_δ sets (see the note to Lemma 4); hence they are residual on U [Lemma 2 (e)]. Let

$$P^* = \bigcap_{K=1}^{\infty} P^K = \{z; \mathscr{D}^* f(z) = +\infty\}.$$

By Lemma 2 (c), P^* is also residual on U; but by the definition of B, $P^* \subset B$, and so B is residual on U [Lemma 2 (g)].

Let $\mathsf{E} = \bigcup_{z \in B} \Psi(z).\mathsf{A}(z)$, and denote by H the interior of the closure of E. Since $|\Psi(z)| \geqslant \pi$, for each $z \in B$, then by Lemma 6 we know that H is not vacuous. We now show that C^* is residual on H. Since $\mathsf{C}^* = \bigcap_{K=1}^{\infty} \mathsf{C}^K$, then by Lemma 2 ($c$) it is sufficient to prove that, for each K, C^K is residual on H. Since, for each K, C^K is a G_δ set, then by Lemma 2 (e) it is sufficient to show that each of the sets C^K is everywhere dense on H. The set E is trivially everywhere dense on H. Let (z_0, θ_0) be a point of $\mathsf{E}.\mathsf{H}$. Since H is open, there is a neighbourhood N of (z_0, θ_0) of the form

$$\{z, \theta; |z - z_0| < \rho, |\theta - \theta_0| < \eta\}$$

which is in H. By the definition of the major arc of tangential unboundedness $\Psi(z_0)$, there is a point (z_1, θ_0) of C^K in the neighbourhood N. Hence C^K is also everywhere dense on H, as desired.

Since C^* is a G_δ set residual on the open set H, we may apply Lemma 5 to obtain a set S residual on U, for all of whose points z_0, $\mathsf{C}^*.\mathsf{A}(z_0)$ is everywhere dense on $\mathsf{H}.\mathsf{A}(z_0)$. Because of the upper semicontinuity of $D^\theta f(z_0)$ (for fixed z_0), the set $\mathsf{C}^*.\mathsf{A}(z_0)$ is closed. Hence for all points z_0 of S, $\mathsf{C}^*.\mathsf{A}(z_0)$ includes the closure of $\mathsf{H}.\mathsf{A}(z_0)$.

Since B is residual on U, then by the second part of Lemma 6 we obtain a set S' also residual on U, for all of whose points z_1 the closure of $\mathsf{H}.\mathsf{A}(z_1)$ includes $\Omega(z_1).\mathsf{A}(z_1)$. Thus at all points z_2 of $S.S' = W$, which is also residual on U,

$$\Omega(z_2).\mathsf{A}(z_2) \subset \mathsf{C}^*.\mathsf{A}(z_2) = \{\theta; D^\theta f(z_2) = +\infty\}.\mathsf{A}(z_2).$$

This is the desired result for upper derivates.

By analogous arguments, using the inequalities (1) of Lemma 3 and the sets $\{z,\theta; D_{\theta+\pi}f(z) \leqslant -K\}$, we obtain another residual set W', at all of whose points z_0 the set $\{\theta; D_{\theta+\pi}f(z_0) = -\infty\}$ includes $\Omega(z_0)$. The set $W^* = W.W'$ is also residual on U, and at all of its points z_1

$$D^{\theta}f(z_1) = -D_{\theta+\pi}f(z_1) = +\infty,$$

for all directions θ in $\Omega(z_1)$.

4. Applications

We now obtain results which are analogues to the one concerning the set G_x defined in § 1.

THEOREM 2. *Let $f(z)$ be a continuous function defined on a plane region R, and let μ be a fixed direction. Then there are two open sets H and J whose union is everywhere dense on R with the following properties: for each point z_0 of H,*

$$D^{\mu}f(z_0) = \partial^{\mu}f(z_0), \qquad D_{\mu+\pi}f(z_0) = \partial_{\mu+\pi}f(z_0);$$

for each point z_1 of a set residual on J,

$$D^{\mu}f(z_1) = D^{\mu+\pi}f(z_1) = +\infty, \qquad D_{\mu}f(z_1) = D_{\mu+\pi}f(z_1) = -\infty.$$

We denote by B the union of the two sets of points $\{z; D^{\mu}f(z) > \partial^{\mu}f(z)\}$ and $\{z; D_{\mu+\pi}f(z) < \partial_{\mu+\pi}f(z)\}$. Let $\bar{B}$ be the closure of B, let $H = R-\bar{B}$, and let $J = R-\bar{H}$. Both H and J are open, their union is everywhere dense on R, and B is everywhere dense on J.

It is clear that H has the properties stated for it; we now prove the stated properties for J. I shall use the terminology of Theorem 1 unless specially noted. For each point z_0 of B, $\Psi(z_0)$ includes one of the arcs $\mu < \theta < \mu+\pi$ or $\mu-\pi < \theta < \mu$. (See the note to Lemma 3.) Denote by B' the subset of B for which $\Psi(z_0)$ includes the former arc, and by B'' the set $B-B'$. Denote by B^* the set $J-\overline{B''}$, and by B_* the set $J-\overline{B^*}$. Both the sets B^* and B_* are open. Their union is everywhere dense on J, and B' and B'' are everywhere dense on B^* and B_* respectively. It will be sufficient to prove the stated properties of the derivates for points z_0 of B^*.

Denote by E the three-dimensional set $\bigcup \Psi(z_0).\mathsf{A}(z_0)$, where the union extends over all points z_0 of $B'.B^*$. In this case the set H (the interior of the closure of E) has an especially simple structure: for each z_0 of B^*, $\mathsf{H}.\mathsf{A}(z_0) = \{\mu < \theta < \mu+\pi\}.\mathsf{A}(z_0)$. Using arguments like those of Theorem 1, we may show first that the sets $\mathsf{C}^* = \{z,\theta; D^{\theta}f(z) = +\infty\}$

and $\mathsf{C}_* = \{z, \theta; D_{\theta+\pi}f(z) = -\infty\}$ are residual on H, and then that there exists a set W residual on B^* such that, for $z_1 \in W$,

$$D^{\theta}f(z_1) = -D_{\theta+\pi}f(z_1) = +\infty, \quad \text{for } \mu \leqslant \theta \leqslant \mu+\pi.$$

From this we have that, for $z_1 \in W$,

$$D^{\mu}f(z_1) = D^{\mu+\pi}f(z_1) = +\infty, \qquad D_{\mu}f(z_1) = D_{\mu+\pi}f(z_1) = -\infty,$$

the desired result.

LEMMA 7. *Let A be a plane G_δ set residual on a region R. Let $[y_0]$ denote a line $y = y_0$. Let B_y denote the set of values y_0 such that the linear set $A.R.[y_0]$ is not everywhere dense on $R.[y_0]$. Then B_y is a linear set of the first category.*

This lemma is essentially the same as Lemma 5, both being special cases of a general result concerning G_δ sets in Cartesian product-spaces.

THEOREM 3. *Let $f(z)$ be a continuous function defined on a plane region R, and let μ be a fixed direction. Then the set of points z_0 such that*

$$\partial^{\mu}f(z_0) \neq -\partial_{\mu+\pi}f(z_0)$$

is a plane set of the first category on R.

It will be sufficient to consider the case $\partial^{\mu}f(z_0) > -\partial_{\mu+\pi}f(z_0)$; the other case is a natural analogue. Let A be the set $\{z; \partial^{\mu}f(z) > -\partial_{\mu+\pi}f(z)\}$, and suppose that A is of the second category. Let $\{r_i\}$ be the rational numbers in the interval $-\infty < x < +\infty$. The set A equals $\bigcup_{r_i > r_j} A_{ij}$, where

$$A_{ij} = \{z; \partial^{\mu}f(z) \geqslant r_i > r_j > -\partial_{\mu+\pi}f(z)\}.$$

Since A_{ij} is of the form $G_\delta . F_\sigma$ (see the note to Lemma 4), then A is a set of the form $(G_\delta . F_\sigma)_\sigma$, or $G_{\delta\sigma}$. If A is of the second category, then at least one of its constituent G_δ sets (call it A^*) is residual on a sub-region R^* of R [Lemmas 2 (b), 2 (a), 2 (e)]. Then by Lemma 7 there exists a line d with direction μ such that the linear G_δ set $A^*.d.R$ is everywhere dense on $d.R^*$, and so residual on $d.R^*$. Letting d have the equation

$$z = z_0 + te^{i\mu},$$

and letting $g(t) = f(z_0 + te^{i\mu})$, we then get the result that the set

$$\{t; D^+g(t) > D^-g(t)\} \supset A^*.d$$

is a linear set of the second category. This is in contradiction to the result stated in § 1, concerning the set G_x. This completes the proof.

THEOREM 4. *Let $f(z)$ be a continuous function defined on a plane region R, and let μ be a fixed direction. Then the set of points z_0 such that*

$$D^{\mu}f(z_0) \neq -D_{\mu+\pi}f(z_0)$$

is a plane set of the first category.

We decompose R into three sets:

$$R = H+J+[R-(H+J)],$$

in the notation of Theorem 2. The third set is a closed set nowhere dense on R, and so we may neglect it. From Theorem 2 we have that the set $\{z; D^{\mu}f(z) = -D_{\mu+\pi}f(z) = +\infty\}$ is residual on J. From Theorems 2 and 3, and the definition of H, we have that the set

$$\{z; z \in H, D^{\mu}f(z) \neq -D_{\mu+\pi}f(z)\}$$

is of the first category on H. Hence the set $\{z; D^{\mu}f(z) \neq -D_{\mu+\pi}f(z)\}$ is of the first category on $J+H$, and thus on R.

5. Examples

There are two natural conjectures concerning further tangential properties of continuous functions $f(z)$. The first concerns the behaviour of $f(z)$ on the set V of Theorem 1, where $f(z)$ is free of tangential singularities. It is that, at all points z_0 of a residual subset of V, $f(z)$ possesses a tangent plane. By this we mean that

$$D^{\theta}f(z_0) = D_{\theta}f(z_0) = \cos\theta . D^0f(z_0)+\sin\theta . D^{\pi/2}f(z_0),$$

for all directions θ. The other concerns the arcs $\Omega(z_1)$ defined at all points z_1 of a residual subset of the set U of Theorem 1. The conjecture is that at all points of this residual set (except perhaps at those of a 'negligible' subset), the arc $\Omega(z_1)$ is either a closed semicircle or a full circle. I show that both of these conjectures are false.

EXAMPLE 1. *There exists a continuous function $f(z)$ defined on the unit square, with uniformly bounded derivates, such that at all points z_1 of a residual set, $D^0f(z_1) = 1$, $D_0f(z_1) = 0$.*

Let E_x be a measurable linear set on the interval $(0, 1)$ such that both E_x and its complement are of positive measure on every sub-interval of $(0, 1)$. Let $\psi_E(x)$ be the characteristic function of E_x, and let

$$h(x) = \int_0^x \psi_E(t)\,dt.$$

Since the derivative to $h(x)$ exists and is equal to $\psi_E(x)$ almost everywhere, then both the sets of points $\{x; D^+h(x) \geqslant 1\}$ and $\{x; D_+h(x) \leqslant 0\}$

are everywhere dense on $(0, 1)$. These are both G_δ sets, hence residual on $(0, 1)$, and so their intersection is also residual on $(0, 1)$; denote this last set by Q_x.

Let
$$f(z) = f(x+iy) = h(x) \quad (0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1).$$

Let Q be the set of vertical lines projecting on Q_x. It is clear that Q is residual on the unit square. It is also easy to see that for any point $z_0 = x_0+iy_0$ of the unit square
$$D^0 f(z_0) = D^+ h(x_0), \qquad D_0 f(z_0) = D_+ h(x_0),$$
and that $\mathscr{D}^* f(z_0) \leqslant 1$, $\mathscr{D}_* f(z_0) \geqslant -1$. But, at any point z_1 of Q,
$$D^0 f(z_1) = 1, \qquad D_0 f(z_1) = 0.$$
Thus $f(z)$ has the stated properties.

EXAMPLE 2. *There exists a continuous function $f(z)$ defined on the unit square S such that, at all points z_2 of a set W^* residual on S,*
$$D^\theta f(z_2) = +\infty, \qquad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \pi,$$
$$D^\theta f(z_2) \leqslant 0, \qquad \pi < \theta < \tfrac{3}{2}\pi.$$

Let $g(z) = g(re^{i\theta})$ be defined as follows. Let
$$g(0) = 0, \qquad g(re^{i\theta}) = g(e^{i\theta}) \quad (r > 1).$$
For $0 < r < 1$, let
$$g(z) = \begin{cases} (r\cos\theta)^{\frac{1}{2}} & (-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi), \\ (r\sin\theta)^{\frac{1}{2}} & (\tfrac{1}{4}\pi \leqslant \theta \leqslant \pi), \\ 0 & (\pi < \theta < \tfrac{3}{2}\pi). \end{cases}$$
Denote by Θ_1 and Θ_2 the arcs $-\frac{1}{2}\pi \leqslant \theta \leqslant \pi$ and $\pi < \theta < \frac{3}{2}\pi$, respectively. We observe that, for $\theta \in \Theta_1$, $D^\theta g(0) = +\infty$ and $D^\theta g(z) \geqslant 0$ for all $z \neq 0$; and for $\theta \in \Theta_2$, $D^\theta g(0) = 0$, and $D^\theta g(z) \leqslant 0$ for all $z \neq 0$. Also, we have $0 \leqslant g(z) \leqslant 1$, for all z.

Let $Z = \{z_j\}$ denote the rational points on S. Let
$$g_j(z) = 2^{-j} g(2^j[z-z_j]).$$
The sum $f(z) = \sum_{j=1}^{\infty} g_j(z)$ is uniformly convergent, and so $f(z)$ is continuous. For any direction $\mu \in \Theta_1$ and any z_j, we have $D^\mu f(z_j) = +\infty$; and for any direction $\nu \in \Theta_2$ for any point $z \in S$, we have $D^\nu f(z) \leqslant 0$. Let $\{\theta_k\}$ be an enumerable set of directions everywhere dense on Θ_1, and let W^k be the set $\{z; D^{\theta_k} f(z) = +\infty\}$. Since W^k includes the set Z, it is

everywhere dense on S, and, being a G_δ set, residual on S. Hence $W^* = \bigcap_{k=1}^{\infty} W^k$ is also residual on S. By the upper semicontinuity of $D^\theta f(z_2)$, we have that, for $z_2 \in W^*$, $D^\theta f(z_2) = +\infty$ for $\theta \in \Theta_1$. Since $D^\theta f(z_2) \leqslant 0$ for $\theta \in \Theta_2$, the construction is complete.

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THE LATTICE DETERMINANTS OF ASYMMETRICAL CONVEX REGIONS (III)

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1. LET K be a convex region in n-dimensional Euclidean space ($n \geqslant 2$) such that K contains the origin O as an interior point. We define the *coefficient of asymmetry* of K as the upper bound λ of the ratio PO/OP', where PP' is an arbitrary chord of K passing through O, and we denote by $V(K)$ and $\Delta(K)$ the (n-dimensional) volume and lattice determinant of K.

In an earlier note† I have obtained an upper bound for the expression $V(K)/\Delta(K)$, where K ranges over all convex regions with coefficient of asymmetry λ. In the present note I consider the same problem when K is restricted to the class of convex regions symmetrical in a point and with coefficient of asymmetry λ. When the centre of symmetry is O, the problem reduces to Minkowski's classical theorem, so that it will be assumed throughout the paper that K is symmetrical in a point other than O.

The main theorem of the paper is

THEOREM 1. *If K is symmetrical in a point and has coefficient of asymmetry λ, then there is a subset L of $K \cup -K$, convex and symmetrical in O, such that*

$$V(L)/V(K) \geqslant q(\lambda) = \begin{cases} \dfrac{2\lambda}{\lambda+1} & (1 \leqslant \lambda \leqslant \lambda_0), \\[2ex] \dfrac{2^n\{1+n(\lambda-1)\}}{(\lambda+1)^n} & (\lambda \geqslant \lambda_0), \end{cases} \tag{1}$$

where λ_0 is the (only) root between $n/(n-1)$ and 2 of the equation

$$\lambda(\lambda+1)^{n-1} = 2^{n-1}(1+n(\lambda-1)).$$

For each λ there is some K and a corresponding L for which the equality sign is needed in (1).

This theorem will be proved in § 5.

† *J. London Math. Soc.* 29 (1954), 251–4.

2. By means of a straightforward application of Minkowski's theorem we deduce from Theorem 1 the theorem:

THEOREM 2_1. *If K is symmetrical in a point and has coefficient of asymmetry λ, then*

$$V(K)/\Delta(K) \leqslant \begin{cases} \dfrac{2^{n-1}(\lambda+1)}{\lambda} & (1 \leqslant \lambda \leqslant \lambda_0), \\ \dfrac{(\lambda+1)^n}{1+n(\lambda-1)} & (\lambda \geqslant \lambda_0). \end{cases}$$

An indication of the accuracy of the upper bound is given by

THEOREM 2_2. *If $1 \leqslant \lambda \leqslant \lambda_0$, there is a K for which*

$$V(K)/\Delta(K) = \frac{2^{n-1}(\lambda+1)}{\lambda}.$$

If $\lambda \geqslant \lambda_0$, there is a K for which

$$V(K)/\Delta(K) \geqslant \frac{(\lambda+1)^n}{n(\lambda-\frac{1}{2})}.$$

Proof of Theorem 2_2. The first statement is substantiated by remarking that the integral lattice is admissible for the parallelepiped K_1 defined by

$$-1/\lambda \leqslant x_1 \leqslant 1, \qquad |x_i| \leqslant 1 \quad (i = 2,..., n),$$

and using the first part of Theorem 2_1.

The second statement is substantiated by considering the region K_2 defined by

$$-1 \leqslant x_1 \leqslant \lambda, \qquad |x_i| \leqslant \tfrac{1}{2}(\lambda+1)-|x_1-\tfrac{1}{2}(\lambda-1)| \quad (i = 2,..., n).$$

This has volume $(\lambda+1)^n/n$ and coefficient of asymmetry λ; and the lattice

$$x_1 = (\lambda-\tfrac{1}{2})u_1$$
$$x_i = \tfrac{1}{2}u_1+u_i \quad (i = 2,..., n)$$

(in which the u's take integral values) is admissible and of determinant $\lambda-\frac{1}{2}$.

From Theorem 2 we deduce as a corollary

THEOREM 3. *When $n = 2$,*

$$V(K)/\Delta(K) \leqslant \begin{cases} 2(\lambda+1)/\lambda & (1 \leqslant \lambda \leqslant 2), \\ (\lambda+1)^2/(2\lambda-1) & (\lambda \geqslant 2). \end{cases}$$

For each value of λ there is some K for which the equality sign is necessary.

3. To avoid interruptions at a later stage, we obtain two straightforward lemmas concerning the function

$$F(\eta) = f(\eta)/g(\eta),$$

where

$$f(\eta) = (1+\alpha\eta)^n-(1-\eta)^n,$$

$$g(\eta) = 1-(1-\eta)^n+2n\alpha\eta.$$

In these expressions α is a positive constant. Since we shall be interested in the function over the interval $0 \leqslant \eta \leqslant 1$, and since $f(0) = g(0) = 0$, we define

$$F(0) = f'(0)/g'(0) = \frac{1+\alpha}{1+2\alpha}.$$

This makes $F(\eta)$ continuous and differentiable on the closed interval.

LEMMA 1. *If* $0 \leqslant \eta \leqslant 1$, *then*

$$F(\eta) \leqslant \max\{F(0), F(1)\}.$$

Proof. We examine first the function

$$G(\eta) = f'(\eta)/g'(\eta).$$

We have

$$G'(\eta) = \frac{g''}{g'}\{H(\eta)-G(\eta)\},$$

where

$$H(\eta) = f''(\eta)/g''(\eta).$$

Now $g''/g' \leqslant 0$ (with equality only when $\eta = 1$ and $n > 2$), so that G is an increasing function when it is greater than H and decreasing otherwise. But

$$H(\eta) = 1-\frac{\alpha^2(1+\alpha\eta)^{n-2}}{(1-\eta)^{n-2}},$$

which is steadily decreasing from $1-\alpha^2$ to $-\infty$. Since G remains finite, it is easy to see that either it is always greater than H, and so increasing, or else it is initially less than H but beyond some point is greater than H; in the latter case G is decreasing in the first part of the interval and increasing in the second part.

To examine the gradient of $F(\eta)$ we now remark that (for $\eta > 0$)

$$F'(\eta) = \frac{g'}{g}\{G(\eta)-F(\eta)\},$$

and, since $g'/g > 0$ for $0 < \eta \leqslant 1$, this shows that F is increasing when it is less than G and decreasing otherwise. Recalling that $F(0) = G(0)$ and noting that for $\eta \neq 0$ we have $F(\eta) = G(\eta)$ only when $F'(\eta) = 0$, we see that, when G is steadily increasing in $0 \leqslant \eta \leqslant 1$, so is F. If G is first decreasing and later increasing, then either F is decreasing over the whole interval or else is first decreasing and later increasing. In any of these cases the assertion of the lemma holds.

LEMMA 2. *We have*

$$\max\{F(0), F(1)\} = \begin{cases} F(0) & (0 < \alpha \leqslant \alpha_0), \\ F(1) & (\alpha \geqslant \alpha_0), \end{cases}$$

the number α_0 being the (only) root between $1/2(n-1)$ and $\frac{1}{2}$ of the equation

$$(1+2\alpha)(1+\alpha)^{n-1}-2n\alpha-1 = 0.$$

Proof. We have

$$F(0) = \frac{1+\alpha}{1+2\alpha}, \qquad F(1) = \frac{(1+\alpha)^n}{1+2n\alpha}.$$

The former is a decreasing function of α for all positive α and the latter is increasing without bound for $\alpha \geqslant 1/2(n-1)$. When $\alpha < 1/2(n-1)$, we observe that

$$F(1) = \frac{(1+\alpha)^n}{1+2n\alpha} > \frac{(1+\alpha)^{n-1}}{2} = G(1).$$

Hence, by the argument of Lemma 1, $F(\eta)$ is decreasing in $0 \leqslant \eta \leqslant 1$ and so $F(1) \leqslant F(0)$. It therefore follows that there is exactly one value of α satisfying

$$F(1) = F(0), \qquad \alpha \geqslant 1/2(n-1),$$

and this solution is not greater than $\frac{1}{2}$ since, when $\alpha = \frac{1}{2}$, we have

$$F(0) = \tfrac{3}{4} \leqslant \frac{3^n}{2^n(n+1)} = F(1).$$

4. To prove Theorem 1 we may suppose without loss of generality that K is strictly convex. We further suppose that the centre of symmetry X of K lies on the x_1-axis at the point $(a, 0,..., 0)$, where $a > 0$. Let the x_1-axis meet the boundary of K in the points A $(-b, 0,..., 0)$, B $(2a+b, 0,..., 0)$, where $b > 0$. We write $a/b = \alpha$, and prove

LEMMA 3. $$\lambda = 1+2\alpha.$$

Proof. Let ω be an arbitrary 2-dimensional plane through the line AB and let POP' be a chord of K in ω. Let θ denote the angle POB and write $\mu(\theta) = PO/OP'$. By the strict convexity of K and by the symmetry of K in X, the tac-lines at P and P' to K which lie in ω intersect on the side of PP' remote from X. Hence μ is decreasing, and so for $0 < \theta \leqslant \pi$,

$$\mu(\theta) < BO/OA = 1+2\alpha.$$

Corollary. Since $\mu(\pi) < 1$, there is a unique chord of K in ω which is bisected at O.

5. Let J denote the sphere

$$x_1 = 0, \qquad x_2^2+...+x_n^2 = 1,$$

and let Y be any point of J. There is a unique 2-dimensional half-plane ω_Y containing Y and bounded by the x_1-axis, and conversely any half-plane bounded by the x_1-axis meets J in a single point. We may express the volume of K as a surface integral over J in the following way. Let $l(r)$ denote the length of the segment in which the line in ω_Y parallel to the x_1-axis and a distance r from it meets K, and write

$$\phi(Y) = \int_0^\infty r^{n-2} l(r)\, dr. \tag{2}$$

Then
$$V(K) = \int_J \phi(Y)\, dS. \tag{3}$$

We now construct the required convex region L. Let CC' be the unique chord bisected at O in the plane of which ω_Y is half, and let C be in ω_Y. As Y varies on J, C traces on the boundary of K a continuous $(n-2)$-dimensional surface Γ which is symmetrical in O. We denote by K^* the part of K lying in the semi-infinite cylinder bounded by generating lines drawn from points of Γ in the positive x_1-direction and by lines joining O to points of Γ. Then K^* and $-K^*$ are non-overlapping. Also, by the symmetry of Γ in O, the convex closure L of $K^* \cup -K^*$ is the union of the convex closures of K^* and $-K^*$. By the convexity of K, these latter are respectively contained in K and $-K$, so that L is contained in $K \cup -K$. Since L contains $K^* \cup -K^*$, it is clearly sufficient, in order to establish Theorem 1, to prove the inequality (1) with $V(K^* \cup -K^*)$ replacing $V(L)$.

Let $s\ [= s(Y)]$ be the distance of C from the x_1-axis. Then, remarking that the line in ω_Y at a distance r $(r < s)$ from the x_1-axis meets $K^* \cup -K^*$ in a segment of length $l(r)+2a$, we have

$$V(K^* \cup -K^*) = \int_J \psi(Y)\, dS, \tag{4}$$

where
$$\psi(Y) = \int_0^s r^{n-2}\{l(r)+2a\}\, dr. \tag{5}$$

Let CD be the chord of K which passes through C and is parallel to the x_1-axis. Then D is the image in X of C', and CD is of length $2a$. Let tac-lines at C, D meet at a point in ω_Y at distance s' from the x_1-axis. We write

$$s' = s(1+\alpha\xi).$$

By the convexity of K we now have the following inequalities,

$$0 \leqslant \xi \leqslant 1,$$

$$\left.\begin{aligned} l(r) &\leqslant 2(a+b) \\ l(r) &\leqslant 2a+\frac{2b}{s\xi}(s-r) \quad (0 \leqslant r \leqslant s(1+\alpha\xi)) \\ l(r) &\geqslant 2a+\frac{2b}{s}(s-r) \quad (0 \leqslant r \leqslant s) \end{aligned}\right\}. \tag{6}$$

We write

$$m(r,\xi) = \min\left\{2(a+b), 2a+\frac{2b}{s\xi}(s-r)\right\}, \tag{7}$$

$$\chi_1(\xi) = \int_0^s r^{n-2}m(r,\xi)\,dr,$$

$$\chi_2(\xi) = \int_s^{s(1+\alpha\xi)} r^{n-2}m(r,\xi)\,dr.$$

Observing that $m(r,\xi)$ is a decreasing function of ξ for $0 \leqslant r \leqslant s$ and that, by (6), (7),

$$\chi_1(1) \leqslant \int_0^s r^{n-2}l(r)\,dr \leqslant \chi_1(\xi),$$

we have, for some η satisfying $\xi \leqslant \eta \leqslant 1$,

$$\int_0^s r^{n-2}l(r)\,dr = \chi_1(\eta).$$

Since $m(r,\xi)$ is an increasing function of ξ for $r \geqslant s$, we have, again using (6), (7),

$$\int_s^\infty r^{n-2}l(r)\,dr \leqslant \chi_2(\xi) \leqslant \chi_2(\eta).$$

Hence by (2), (5), and by performing the integration, we obtain

$$\frac{\phi(Y)}{\psi(Y)} \leqslant \frac{\chi_1(\eta)+\chi_2(\eta)}{\chi_1(\eta)+2as^{n-1}/(n-1)} = F(\eta).$$

Using Lemmas 1, 2, 3, we therefore have

$$\psi(Y)/\phi(Y) \geqslant q(\lambda),$$

and so, by (3), (4),

$$V(K^* \cup -K^*)/V(K) \geqslant q(\lambda).$$

To exhibit regions K and L for which equality holds in (1) we use the regions K_1, K_2 of Theorem 2_2. For K_1 the corresponding L is the parallelepiped

$$|x_i| \leqslant 1 \quad (i = 1, 2, \ldots, n).$$

For K_2 the corresponding L is the region

$$|x_1| \leqslant \lambda, \qquad |x_i| \leqslant \min\{1, \lambda - |x_1|\} \quad (i = 2, \ldots, n).$$

This completes the proof of Theorem 1.

ON THE COHOMOLOGY GROUPS OF A FINITE GROUP

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1. Introduction

LET Q be a multiplicative group and G an additive abelian group admitting Q as a group of left operators. The n-dimensional cohomology groups $H^n(Q, G)$ $(n \geqslant 0)$ are defined by S. Eilenberg and S. Maclane in (**1**). In this paper Q will be a finite group.

Let F be any field, A the group algebra of Q over F, and M any two-sided A-module; since A has a finite basis, the cohomology groups $H^n(A, M)$ are also defined for $n \geqslant 0$ (**3**). In case of ambiguity the group algebra over F will be written $A\{F\}$.

I shall first consider under what conditions G can be isomorphically embedded in a two-sided $A\{F\}$-module for a suitable choice of F and what is the relation between the groups $H^n(Q, G)$ and $H^n(A\{F\}, M)$. For this purpose Q may be supposed to operate on G identically from the right. Theorem 1 gives necessary and sufficient conditions on G that it may be taken as M; if G has only elements of infinite order but does not satisfy the condition of Theorem 1, F can be taken as the field R of rational numbers and M as the tensor product $R \circ G$ of the additive group of R and G (**7**). In this case the group of n-cocycles $Z^n(A, M)$ is isomorphic to the tensor product of R and the group $Z^n(Q, G)$ of cocycles on Q to G.

The second problem considered is the structure of $H^n(Q, G)$ for various types of coefficient group G. The results of the first theorem and lemma can be combined with a result due to G. P. Hochschild (**3**) on the cohomology groups of a separable algebra, that $H^n(A, M) = 0$ $(n \geqslant 1)$ for every two-sided A-module M, to deduce the vanishing of $H^n(Q, G)$ $(n \geqslant 1)$ when G satisfies certain conditions (Theorem 2).

When all the elements of G have finite order, further results are obtained by using direct methods. In § 4 all the elements of G have order p^k, where p is a prime, and Theorem 4 shows that $H^n(Q, G) = 0$ $(n \geqslant 1)$ if p is prime to the order of Q. In § 5 the orders of the elements of G are finite and $H^n(Q, G)$ is found to be the direct sum of groups $H^n(Q, G_i)$ when G_i is the subgroup of elements of order p_i^k for any integer k and a prime divisor p_i of the order of Q. When k is bounded for each i, Theorem 6 gives a maximum to the order of $H^n(Q, G)$; three examples

are given to show that in some, but not all, cases the order is less than this maximum.

2. A-Modules

LEMMA 1. *If, for some field F, G is a two-sided A-module M, then*

$$H^n(Q, G) \approx H^n(A, M).$$

is an isomorphism.

Since the elements of Q are the basis elements of A, given $f \in C^n(A, M)$ (the group of n-cochains on A with values in M) we can define $\lambda f \in C^n(Q, G)$ by taking $\lambda f = f$ restricted to the basis elements of A.

Since we can assume that the identity of A acts as the identity operator on M (**4**), the same is true of G. The elements of Q, considered as elements of A, operate on G and, if the coboundary operator δ on $C^n(Q, G)$ is defined by means of these induced operators, then $\lambda\delta = \delta\lambda$. Hence λ is a cochain-isomorphism into $C^n(Q, G)$.

Conversely, any $f \in C^n(Q, G)$ can be extended to $\lambda^{-1}f \in C^n(A, M)$ by linearity, since the condition on G implies that A can operate on G. Therefore λ is an isomorphism onto and induces

$$H^n(A, M) \approx H^n(Q, G).$$

I shall consider when the above condition can be satisfied.

THEOREM 1. *A necessary and sufficient condition that G can be a two-sided A-module for some field F is that either* (i) *G has no elements of finite order and is completely divisible,*† *or* (ii) *every element of G has order a prime p.*

Any field F contains as prime subfield either the field R of rationals or the field J_p of integers *modulo* p. If G is an A-module, then G admits F, and with it either R or J_p, as a ring of operators.

Condition (i) is necessary and sufficient that G admit R as a ring of operators [(**7**) Theorem 15]. If J_p operates on G, then, for any $g \in G$, $pg = Og = 0$. The conditions are therefore necessary.

Suppose now that (i) is satisfied; then R operates on G in the natural way. Take F to be R and let A be the group algebra of Q over R. Define

$$\Big(\sum_i r_i q_i\Big) \cdot g = \sum_i r_i(q_i \cdot g),$$

$$g \cdot \Big(\sum_i r_i q_i\Big) = \sum_i r_i(g \cdot q_i),$$

$$(sq) \cdot (rg) = sr(q \cdot g) = (rq) \cdot (sg),$$

$$(rg) \cdot (sq) = rs(g \cdot q) = (sg) \cdot (rq), \tag{2.1}$$

† i.e., given any integer d and any $g \in G$, there is an element $g' \in G$ such that $dg' = g$.

where $s, r_i \in R, g \in G$ and the summation extends over all $q_i \in Q$. Here, $g.q = g$ according to the convention of § 1, but the right operation by q is exhibited in order that right operators $\sum rq \in A$ should satisfy the linearity condition (2.2)(c).

This makes G an A-module; for

$$a.(g+g') = a.g+a.g', \qquad (g+g').a = g.a+g'.a \qquad (2.2)(a)$$

is clearly satisfied for every $a \in A$ since it is satisfied for $a \in Q$. Also

$$\begin{aligned}\Big(\sum_i r_i q_i \sum s_j q_j\Big).g &= \Big(\sum_{i,j} r_i s_j q_i q_j\Big).g \\ &= \sum_{i,j} r_i s_j(q_i q_j.g) \\ &= \Big(\sum_i r_i q_i\Big).\Big\{\sum_j s_j(q_j.g)\Big\} \\ &= \Big(\sum_i r_i q_i\Big).\Big\{\Big(\sum_j s_j q_j\Big).g\Big\},\end{aligned}$$

and similarly

$$a.(g.a') = (a.g).a', \qquad g.(aa') = (g.a).a' \qquad (2.2)(b)$$

$$(a+a').g = a.g+a'.g, \qquad g.(a+a') = g.a+g.a', \qquad (2.2)(c)$$

where $a, a' \in A$ $(g \in G)$.

If condition (ii) is satisfied in place of (i), J_p can operate on G in the natural way for some prime p. In this case take F to be J_p and define operators from A on Q as in (2.1) with $r_i, s \in J_p$. As before this makes G an A-module.

Lemma 2. *Let G_i $(i = 1, 2,...)$ be abelian groups such that the order of every element of G_i divides n_i, and the integers n_i are mutually prime; also let $G = \sum_i G_i$, a direct sum. Then Q operates on each G_i, and*

$$H^n(Q, G) \approx \sum_i H^n(Q, G_i).$$

If the number of summands G_i is infinite, an element of G will be a sum $\sum_i g_i$ with only a finite number of the g_i non-zero.

Let $q \in Q$ and $g_i \in G_i$; then $q.g_i$ can be expressed as a finite sum

$$q.g_i = g'_{j_1}+g'_{j_2}+...+g'_{j_k}$$

with $g'_{j_r} \in G_{j_r}$ for $r = 1,..., k$. The condition on the orders n_i implies that $g'_{j_r} = 0$ unless $j_r = i$; each summand G_i therefore admits operators from Q induced by the operators on G.

Given any $f \in C^n(Q, G)$, define $f_i \in C^n(Q, G_i)$ by

$$f(q_1,..., q_n) = f_{i_1}(q_1,..., q_n)+...+f_{i_k}(q_1,..., q_n).$$

Since Q is finite, there are, for each f, only a finite number of suffixes i_r for which $f_{i_r}(q_1,..., q_n)$ is not zero for all choices of $q_1,..., q_n \in Q$. Hence

$$f = \sum_{r=1}^{k} f_{i_r}$$

is a finite sum. Also, since operators from Q on G induce operators on each G_i, $(\delta f)_i = \delta f_i$ and the lemma follows.

THEOREM 2. *Let either*

(i) *G be completely divisible, without elements of finite order, or*

(ii) *$G = \sum_i G_i$, a direct sum, where every element of G_i has for order a prime p_i which does not divide the order of Q, and $p_i \neq p_j$ if $i \neq j$.*

Then $H^n(Q, G) = 0$ $(n \geqslant 1)$.

If the characteristic of F does not divide the order of Q, the group algebra A is separable and $H^n(A, M) = 0$ $(n \geqslant 1)$ for any two-sided A-module M (**3**). It follows from Lemma 1 that, if G is an A-module, then $H^n(Q, G) = 0$ $(n \geqslant 1)$. Theorem 1 states that condition (i) is sufficient for this. When condition (ii) is satisfied, it follows from Theorem 1 (ii) that each G_i can be made into an $A\{J_{p_i}\}$-module. Since each p_i is prime to the order of Q, each of the algebras $A\{J_{p_i}\}$ is separable and therefore $H^n(Q, G_i) = 0$ $(n \geqslant 1)$ for each i. By Lemma 2 this implies that $H^n(Q, G) = 0$.

3. Coefficient groups without elements of finite order

In this section I shall assume that G has no elements of finite order but is not necessarily completely divisible. As in Theorem 1 (i), A will be the group algebra of Q over the field R of rational numbers, but it will not in general be possible to take M equal to G (Theorem 1).

I shall consider instead the tensor product (**7**) $R \circ G$ of the field of rationals (considered as an additive group) and G. Elements of $R \circ G$ will be written $r \circ g$ with $r \in R$, $g \in G$, and satisfy the two distributive laws

$$r \circ (g+g') = r \circ g + r \circ g', \qquad (r+r') \circ g = r \circ g + r' \circ g. \tag{3.1}$$

Since G has no elements of finite order, it can be embedded isomorphically in $R \circ G$ by identifying $1 \circ g$ with g. Any element of $R \circ G$ can be written uniquely in normal form, i.e.

$$\frac{a}{b} \circ g = \frac{1}{b} \circ ag = \frac{1}{b} \circ g'$$

for a, $b \in J$ (the rational integers) and g, $g' \in G$.

R operates on $R \circ G$ by writing

$$r'.(r \circ g) = r'r \circ g = (r \circ g).r', \tag{3.2}$$

for, since G has no elements of finite order, $R \circ G$ has unique division, i.e., if

$$\frac{1}{b} \circ g = \frac{1}{b} \circ g', \quad \text{then } g = g'.$$

In fact $R \circ G$ is the smallest completely divisible group containing G [(**7**) Theorem 21], and therefore by Theorem 1 (i) is the smallest group containing G which can be taken as an A-module. That we can take $M = R \circ G$ will follow as in Theorem 1 if Q operates on M. Define

$$q.(r \circ g + r' \circ g' + \ldots) = r \circ (q.g) + r' \circ (q.g') + \ldots,$$
$$(r \circ g + r' \circ g' + \ldots).q = r \circ (g.q) + r' \circ (g'.q) + \ldots. \tag{3.3}$$

Then conditions similar to (2.2) (a) and (b) with Q in place of A are clearly satisfied for M since they are satisfied for G, and Q operates on M.

The operators from A on M are now defined by (3.2), (3.3), and (2.1) with M in place of G.

Given any $f \in C^n(Q, G)$ we can define $\sigma_d f \in C^n(A, M)$ by

$$(\sigma_d f)(q_1,\ldots,q_n) = \frac{1}{d} \circ f(q_1,\ldots,q_n) \tag{3.4}$$

and by linearity, where $q_i \in Q$, $d \in J$.

Lemma 3. *σ_d: $C^n(Q, G) \to C^n(A, M)$ is an isomorphism (into) for each $d \in J$, and $\delta\sigma_d = \sigma_d \delta$.*

I shall prove the lemma for the case $d = 1$, but the method of proof holds for any $d \in J$.

Now σ_1 is clearly an isomorphism, for $\sigma_1 f = 0$ implies $1 \circ f(q_1,\ldots,q_n) = 0$ for all q_i; therefore $f(q_1,\ldots,q_n) = 0$ for all q_i and $f = 0$. Also

$$\begin{aligned}(\delta\sigma_1 f)(q_1,\ldots,q_{n+1}) &= q_1.(\sigma_1 f)(q_2,\ldots,q_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i (\sigma_1 f)(q_1,\ldots,q_i q_{i+1},\ldots,q_{n+1}) \\ &\quad + (-1)^{n+1}(\sigma_1 f)(q_1,\ldots,q_n).q_{n+1} \\ &= 1 \circ (\delta f)(q_1,\ldots,q_{n+1}) \\ &= (\sigma_1 \delta f)(q_1,\ldots,q_{n+1}),\end{aligned}$$

when we use the definition of operators (3.3) and the distributive law (3.1).

Let $C^n_d(A, M)$ be the subgroup of $C^n(A, M)$ consisting of cochains f^*, where, for all $q_i \in Q$,

$$f^*(q_1,\ldots,q_n) = \frac{1}{d} \circ g$$

for fixed $d \in J$ and some $g \in G$. Then

$$\sigma_d\colon C^n(Q, G) \to C^n_d(A, M)$$

is an isomorphism onto and induces an isomorphism

$$\sigma_d\colon H^n(Q, G) \approx H^n_d(A, M).$$

LEMMA 4. $Z^n(A, M) \approx R \circ Z^n(Q, G)$.

Since G has no non-zero elements of finite order, neither has $C^n(Q, G)$. For, if m is any integer, $mf = 0$ for $f \in C^n(Q, G)$ implies $mf(q_1, \ldots, q_n) = 0$ for all q_i. But then $f(q_1, \ldots, q_n) = 0$, $f = 0$. $C^n(Q, G)$ may therefore be embedded isomorphically in $R \circ C^n(Q, G)$.

Given $f \in C^n(Q, G)$, define

$$\lambda\colon R \circ C^n(Q, G) \to C^n(A, M)$$

by
$$\lambda\left(\frac{1}{d} \circ f\right) = \sigma_d f \in C^n(A, M).$$

Here λ is an isomorphism into since σ_d is an isomorphism into for each d and $\frac{1}{d} \circ 0 = 0$.

Let $f^* \in C^n(A, M)$ be defined by its value on the basis elements of A,

$$f^*(q_{i_1}, \ldots, q_{i_n}) = \frac{1}{d_{i_1 \ldots i_n}} \circ g_{i_1 \ldots i_n},$$

where $d_{i_1 \ldots i_n} \in J$, $g_{i_1 \ldots i_n} \in G$. Since there are only a finite number of such elements, the $d_{i_1 \ldots i_n}$ have an L.C.M. b (say). Let $d_{i_1 \ldots i_n} b_{i_1 \ldots i_n} = b$, and let $f \in C^n(Q, G)$ be defined by

$$f(q_{i_1}, \ldots, q_{i_n}) = b_{i_1 \ldots i_n} g_{i_1 \ldots i_n}.$$

Then
$$f^* = \lambda\left(\frac{1}{b} \circ f\right).$$

Hence λ is onto, and

$$C^n(A, M) \approx R \circ C^n(Q, G).$$

Also, by Lemma 3,

$$\delta f^* = \delta \sigma_b f = \sigma_b \delta f = \lambda\left(\frac{1}{b} \circ \delta f\right),$$

and $\delta f^* = 0$ if and only if $\delta f = 0$; therefore λ induces an isomorphism

$$R \circ Z^n(Q, G) \approx Z^n(A, M).$$

LEMMA 5. *Let $f^* = \delta h^* \in B^n(A, M)$, and let $f \in C^n(Q, G)$ be defined as in Lemma 4. There is an integer d such that $df \in B^n(Q, G)$.*

As in the preceding lemma, $f \in C^n(Q, G)$ may be defined by the

equation $f^* = \sigma_b f$ for some integer b, and $h^* = \sigma_d h$ for some $d \in J$, $h \in C^{n-1}(Q, G)$. By Lemma 3, $\delta h^* = \delta\sigma_d h = \sigma_d \delta h$. Hence

$$\frac{1}{b} \circ f = \frac{1}{d} \circ \delta h,$$

and multiplication by bd gives $df = b\delta h = \delta bh \in B^n(Q, G)$.

Corollary. *Every element of $H^n(Q, G)$ $(n \geqslant 1)$ has finite order.*

With any $f \in Z^n(Q, G)$ associate $\sigma_1 f \in Z^n(A, M)$. Since A is separable, $H^n(A, M) = 0$ $(n \geqslant 1)$, and $\sigma_1 f = \delta h^*$ for some $h^* \in C^{n-1}(A, M)$. It follows from the lemma that $df = \delta k$ for some integer d and some $k \in C^{n-1}(Q, G)$; hence $d\{f\} = \{df\} = 0$, where $\{f\} \in H^n(Q, G)$ is the class containing f.

This result is a particular case of the more explicit result of Theorem 3, which is proved in the next section by direct methods for any abelian coefficient group.

If we allow G to have elements of both finite and infinite order, but demand that, if the order is finite, it be prime to the order of Q, it follows from Theorem 5 Corollary 1 below that the groups $H^n(Q, G)$ reduce to those considered above. For let G' be the set of all $g \in G$ such that $mg = 0$ for some integer m; then G' is a subgroup of G such that $qG' = G'$ for any $q \in Q$, and Q operates on G'. The factor group $G - G'$ has no elements of finite order and also admits Q as a group of operators according to the rule $q\bar{g} = \overline{qg}$, where $\bar{g}$ is the coset of G containing g. If G' contains not all the elements of finite order, but those with order prime to that of Q, we have

Lemma 6. *Let G' be the subgroup of G containing all the elements of finite order prime to the order of Q. Then*

$$H^n(Q, G) \approx H^n(Q, G - G') \quad (n \geqslant 1).$$

The homomorphisms in the exact sequence

$$0 \to G' \xrightarrow{i} G \xrightarrow{j} G - G' \to 0$$

are all operator homomorphisms, and we obtain the exact sequence (5)

$$\to H^n(Q, G') \xrightarrow{i^*} H^n(Q, G) \xrightarrow{j^*} H^n(Q, G - G') \xrightarrow{\delta} H^{n+1}(Q, G') \to. \quad (3.5)$$

The condition on G' is that of Theorem 5 Corollary 1 below, and so $H^n(Q, G') = 0$ $(n \geqslant 1)$; this implies the isomorphism of the lemma.

4. Coefficient groups whose elements have prime-power order

In this section G will be a group all of whose elements have order p^k for a given prime p and some integer k.

I shall first prove the following formal generalization of a known theorem to the case when Q operates non-trivially on any abelian group G [the cases $n = 1$ and 2 with non-trivial operators are proved by Zassenhaus (**8**), 131–2, Theorems 24, 26].

THEOREM 3. *Let G be any abelian group and Q have finite order t. Then $tH^n(Q, G) = 0$ $(n \geqslant 1)$.*

Suppose first that Q operates identically from the right. For any $f \in C^n(Q, G)$ define $f^* \in C^{n-1}(Q, G)$ by

$$f^*(q_1,\dots, q_{n-1}) = (-1)^n \sum_{q\in Q} f(q_1,\dots, q_{n-1}, q). \tag{4.1}$$

Then

$$\begin{aligned}(-1)^n(\delta f^*)(q_1,\dots, q_n) &= q_1 \sum_{q\in Q} f(q_2,\dots, q_n, q)+\\ &\qquad +\sum_{i=1}^{n-1}(-1)^i \sum_{q\in Q} f(q_1,\dots, q_i q_{i+1},\dots, q_n, q)+\\ &\qquad +(-1)^n \sum_{q\in Q} f(q_1,\dots, q_{n-1}, q)\\ &= \sum_{q\in Q} (\delta f)(q_1,\dots, q_n, q)+(-1)^n \sum_{q\in Q} f(q_1,\dots, q_n)q.\end{aligned}$$

If $\delta f = 0$, this implies that

$$tf(q_1,\dots, q_n) = \sum_{q\in Q} f(q_1,\dots, q_n)q = (\delta f^*)(q_1,\dots, q_n),$$

i.e. $t\{f\} = \{tf\} = 0$.

In the case when Q operates identically from the left the correspondence is

$$f'(q_1,\dots, q_{n-1}) = \sum_{q\in Q} f(q, q_1,\dots, q_{n-1}).$$

It is shown by Eilenberg and Maclane (**1**) that, if Q operates from both sides, the cohomology group is isomorphic to the one obtained from a suitable set of left operators; the theorem is therefore true in this case also.

THEOREM 4. *Let the order of every element of G be a power of the prime p, and let Q have order t.*

(i) *If $(p,t) = 1$, $H^n(Q, G) = 0$ $(n \geqslant 1)$;*

(ii) *If there is a least integer r such that the order of every $g \in G$ is p^k with $k \leqslant r$ and if $(p^r,t) = p^s$, then $p^sH^n(Q, G) = 0$ $(n \geqslant 1)$.*

Any $\{f\} \in H^n(Q, G)$ has order p^m for some integer m. For the cocycle f determines only a finite number of elements $f(q_1, q_2,\dots, q_n) \in G$, where the q_i take all values in Q; since the order of each of these elements is a power $k_{q_1\dots q_n}$ of p, therefore $p^mf = 0$, where m is the maximum of the integers $k_{q_1\dots q_n}$, and $p^m\{f\} = \{p^mf\} = 0$.

If $(p,t) = 1$, then $(p^m,t) = 1$ and there are integers a, b such that $ap^m+bt = 1$. It follows from Theorem 3 that

$$\{f\} = ap^m\{f\}+bt\{f\} = 0.$$

In the second part of the theorem, if $s = r$, the result follows from the above, for in this case $m = r$ since every $k_{q_1\dots q_n} \leqslant r$. Let $s < r$ and $t = p^s u$, where $(p,u) = 1$. Then $(p^{r-s},u) = 1$ and there are integers a, b such that $ap^{r-s}+bu = 1$. If $\{f\} \in H^n(Q,G)$, since $p^r\{f\} = 0$, therefore, using Theorem 3 again, we get

$$p^s\{f\} = ap^r\{f\}+bp^s u\{f\} = 0.$$

5. Coefficient groups whose elements have finite order

When all the elements of G have finite order, we can combine the results of Lemma 2 and Theorem 4 (i) to reduce the calculation of $H^n(Q,G)$ to the calculation of $H^n(Q,G_i)$, where each G_i is a subgroup of G satisfying the conditions of the last section for some prime p_i.

THEOREM 5. *Let the order of every element of G be finite, and let G_i be the subgroup of all the elements of G whose order is a power of the prime p_i. Then*

$$H^n(Q,G) \approx H^n(Q,G_{j_1})+\dots+H^n(Q,G_{j_m}) \quad (n \geqslant 1),$$

a direct sum, where p_α ($\alpha = j_1,\dots,j_m$) are the distinct prime divisors of the order of Q.

Any element $g \in G$ is of finite order $m = p_1^{\rho}\dots p_r^{\sigma}$, where the p_i are distinct primes, and is therefore expressible uniquely as a finite sum $g = \sum_{i=1}^{r} g_i$ with $g_i \in G_i$. It follows that G is a direct sum $G = \sum_i G_i$, where the group G_i is empty if no non-zero element of G has order divisible by p_i. We can apply Lemma 2 and obtain

$$H^n(Q,G) \approx \sum_i H^n(Q,G_i),$$

a direct sum.

If $i \neq j_k$ for some k, p_i is prime to the order of Q and by Theorem 4 (i), $H^n(Q,G_i) = 0$. It follows that the only non-zero groups in the expression for $H^n(Q,G)$ are those for which p_i divides the order of Q.

COROLLARY 1. *If the order of every element of G is prime to the order of Q, then $H^n(Q,G) = 0$ ($n \geqslant 1$).*

COROLLARY 2. *Let G satisfy the condition of Corollary 1. Every extension of Q by G splits over G.*

By (1) 3.2 the extensions of Q by G are in one-one correspondence

with the elements of $H^2(Q, G)$; an extension of Q by G splits over G if and only if the corresponding element of $H^2(Q, G)$ is zero.

COROLLARY 3. *Let the centre G of a group K satisfy the condition of Corollary 1. The Q-kernel (K, θ) is extendible for all θ, and there is only one equivalence class of extensions of Q by (K, θ).*

The condition that (K, θ) be extendible is that the invariant

$$\{k\} \in H^3(Q, G)$$

vanish [(**2**) Theorem 8.1], and this is satisfied for any θ.

In this case the non-equivalent extensions of Q by (K, θ) are in one–one correspondence with the elements of $H^2(Q, G)$. Since there is only one such element, it follows that there is, within equivalence, one and only one extension.

When G is a finite group, Corollary 2 is contained in a known theorem [(**8**) 132, Theorem 25], which states:

If the order of K is prime to the order of Q, every extension of Q by K splits over K.

This is a more precise result, using a stronger condition, than that of Corollary 3.

If, in addition, each G_i satisfies the condition of Theorem 4 (ii), we can combine the results of that theorem and of Theorem 5 to obtain a stronger form of Theorem 3.

THEOREM 6. *Let every element of G have finite order; let p_i $(i = 1,..., m)$ be the distinct prime divisors of the order t of Q and let G_i be the subgroup of G containing all the elements of order p_i^k. If, for each i, there exists a least integer r_i such that $k \leqslant r_i$ for all the elements of G_i and if $(e, t) = d$, where $e = p_1^{r_1}...p_m^{r_m}$, then*

$$dH^n(Q, G) = 0.$$

The result is an immediate consequence of the two theorems.

6. A lemma on crossed homomorphisms

In the next section I shall calculate the value of $H^1(Q, G)$ for a particular choice of Q and G; in this calculation it is necessary to determine the value of $Z^1(Q, G)$, i.e. the group of crossed homomorphisms on Q to G, and I therefore prove the following lemma.

LEMMA 7. *Let Q be a finitely generated multiplicative group operating on an additive group G. Let the generators of Q be $q_1,..., q_n$ subject to a set T of defining relations $t(q) = 1$. A crossed homomorphism f: $Q \to G$*

is uniquely determined by its values on the generators q_i, *provided that* f *is consistent with the relations* T; *i.e.*

$$f(t) = 0 \quad \textit{for every } t \in T.$$

Let $f(q_i) = g_i \in G$. Any element $q \in Q$ is expressible (not uniquely) as a reduced word

$$w(q) = q_{i_1}^{\epsilon_1} q_{i_2}^{\epsilon_2} \ldots q_{i_r}^{\epsilon_r} \quad (\epsilon_j = \pm 1), \tag{6.1}$$

where we can assume that, if $i_j = i_{j+1}$, then $\epsilon_j = \epsilon_{j+1}$, i.e. no cancelling is possible; if $q = 1$, the word may be the empty word, i.e. $w(1) = 1$. In particular each $t(q) \in T$ is a word of this form.

Let $f\{w(q)\}$ be defined by

$$f(1) = 0, \qquad f\{w(q)\} = \epsilon_1 x_1 g_{i_1} + \epsilon_2 x_2 g_{i_2} + \ldots + \epsilon_r x_r g_{i_r}, \tag{6.2}$$

where $$x_j = q_{i_1}^{\epsilon_1} \ldots q_{i_{j-1}}^{\epsilon_{j-1}} q_{i_j}^{\delta} \quad \begin{cases} \delta = 0 & \text{if } \epsilon_j = +1, \\ \delta = \epsilon_j & \text{if } \epsilon_j = -1, \end{cases}$$

for all $j \geqslant 1$ and $q_i^0 \equiv 1$.

Then $f(q_i) = q_i^0 g_i = g_i$, $f(q_i^{-1}) = -q_i^{-1} g_i$. Also

$$\begin{aligned} f\{w(q) q_{i_{r+1}}^{\epsilon_{r+1}}\} &= \epsilon_1 x_1 g_{i_1} + \ldots + \epsilon_r x_r g_{i_r} + \epsilon_{r+1} q_{i_1}^{\epsilon_1} \ldots q_{i_r}^{\epsilon_r} q_{i_{r+1}}^{\delta} g_{i_{r+1}} \\ &= f\{w(q)\} + q_{i_1}^{\epsilon_1} \ldots q_{i_r}^{\epsilon_r} (\epsilon_{r+1} q_{i_{r+1}}^{\delta} g_{i_{r+1}}) \\ &= f\{w(q)\} + w(q) f\{q_{i_{r+1}}^{\epsilon_{r+1}}\}. \end{aligned}$$

It follows by induction on r that, for any two words w, w',

$$f(ww') = f(w) + w f(w') \tag{6.3}$$

is satisfied [cf. (**6**) proof of Lemma 3], where any word w is supposed to operate on G in the same way as the element of Q which it represents; in particular any word $t \in T$ will be the identical operator on G.

Suppose now that the elements $g_i \in G$ are chosen so that, if $f(t)$ is expressed in the form (6.2) for each $t \in T$, the right-hand side of each expression is zero. Then for any word w and any $t \in T$,

$$\begin{aligned} f(wtw^{-1}) &= f(w) + w f(t) + wt f(w^{-1}) \\ &= f(w) + w f(w^{-1}) \\ &= 0. \end{aligned}$$

Also if w, w' are two words such that $f(w) = 0 = f(w')$, then

$$f(ww') = f(w) + w f(w') = 0.$$

Since any relation $\tau = 1$ between the generators is obtained as a product of transforms of words $t \in T$, it follows by repeated application

of these two relations that $f(\tau) = 0$ whenever $\tau = 1$. Also, if w, w', τ are words such that $\tau = 1$, then

$$\begin{aligned} f(w\tau w') &= f(w)+wf(\tau)+w\tau f(w') \\ &= f(w)+wf(w') \\ &= f(ww') \end{aligned}$$

since τ represents the identity of Q and therefore operates identically on G.

Let w, w' be two reduced words representing the same element $q \in Q$. Then w' can be obtained from w by inserting or removing words $\tau = 1$, and by the above relations $f(w') = f(w)$. Hence f is a single-valued map of Q to G and by (6.3) is a crossed homomorphism.

Now let ϕ be a crossed homomorphism $Q \to G$ such that $\phi(q_i) = g_i$. Repeated application of (6.3) shows that, if w is given by (6.1), then $\phi(w)$ is given by (6.2). Also, since $\phi(1) = 0$, $\phi(t) = 0$ for every $t \in T$; therefore ϕ agrees with f over the whole of Q.

7. Three examples

The following three examples show that the order of $H^n(Q, G)$ can be less than the maximum found in Theorem 4 (ii); the calculation in Example 2 is carried out for $n = 1$ and in all cases the notation is that of the theorem.

Example 1. Let Q be cyclic of order p^m and G be cyclic of order p^r, where p is prime, and let Q operate identically on G. Then $H^n(Q, G)$ is cyclic of order p^s, where $s = \min(m, r)$.

The value of $H^n(Q, G)$ has been calculated by Eilenberg and Maclane (1) when Q is cyclic and operates identically on G as follows:

$$H^{2n+1}(Q, G) = G(p^m), \qquad H^{2n+2}(Q, G) = G-p^mG \quad (n \geqslant 0)$$

where $G(p^m)$ consists of the elements of G for which $p^mg = 0$.

If $r \leqslant m$, then $s = r$. Also $G(p^m) = G$ and $p^mG = 0$; hence

$$H^n(Q, G) = G \quad (\text{all } n).$$

If $r > m$, then $G(p^m)$ is the cyclic group $\{p^{r-m}g\}$ generated by $p^{r-m}g$, where g is a generator of G, and $s = m$. In this case

$$H^{2n+1}(Q, G) = \{p^{r-m}g\}, \qquad H^{2n+2}(Q, G) = G-\{p^mg\}.$$

In both cases $p^kH^n(Q, G) = 0$ implies $k \geqslant s$, and the order of $H^n(Q, G)$ is equal to the maximum value.

Example 2. Let Q be the quaternion group, and G be cyclic of order 8. Then $p^r = p^s = 8$ but, with a certain choice of operators, the order of $H^1(Q, G)$ is 4.

Q can be generated by elements a and b with relations

$$a^4 = 1, \qquad b^2 = a^2, \qquad ba = a^3 b. \tag{7.1}$$

A generator of G will be denoted by g.

As in the last section any cocycle $f \in Z^1(Q, G)$ is defined by its values on a and b, provided that these are consistent with the relations (7.1).

Let Q operate on G from the left according to the rule

$$a.g = b.g = 3g. \tag{7.2}$$

The relations (7.1) imply the following conditions on f:

$$\begin{aligned} (1+a+a^2+a^3)f(a) &= 0, \\ (1+a)f(a) &= (1+b)f(b), \\ (1+a+a^2)f(a)+a^3f(b) &= f(b)+bf(a). \end{aligned} \tag{7.3}$$

Of these the first is satisfied identically and the second is a consequence of the third, which reduces to

$$2\{f(a)+f(b)\} = 0;$$

hence $f(b) = 3f(a)+4kg$ $(k = 0, 1)$ and $Z^1(Q, G)$ has 16 elements.

If $f = \delta(hg)$ for some integer h, then

$$f(a) = f(b) = 2hg,$$

giving 4 possible values. $H^1(Q, G)$ has, therefore, 4 elements, which can be represented by the cocycles f_i $(i = 0, 1, 2, 3)$, where

$$f_i(a) = \begin{cases} 0 & (i = 0, 2), \\ g & (i = 1, 3), \end{cases}$$

$$f_i(b) = \begin{cases} 3f_i(a) & (i = 0, 1), \\ 3f_i(a)+4g & (i = 2, 3). \end{cases}$$

This is a cyclic group of order 4 generated by $\{f_1\}$.

Example 3. Let Q be the group of two elements, 1 and a, and let G be cyclic of order 2^m $(m > 2)$ generated by an element g. Let

$$a.g = (2^{m-1}+1)g.$$

Using the result of Eilenberg and Maclane [(**1**) 16.2, 16.3] we have in this case

$$H^n(Q, G) = 0 \quad (n \geqslant 1)$$

although the orders of Q and G are not relatively prime.

I should like to thank Professor Whitehead for his help in the preparation of this paper.

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GEGENBAUER TRANSFORMS

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1. Introduction

SEVERAL authors (**3**, **4**) have recently introduced and shown the usefulness of Legendre and Jacobi transforms. In particular, Churchill and Dolph (**1**) have obtained a convolution property for Legendre transforms which expresses the inverse transform of the product of two transforms directly in terms of the two object functions. In this note I define the Gegenbauer transform and obtain some of its properties, including a generalization of the convolution property for Legendre transforms.

2. Definition and properties of the Gegenbauer transform

I define the Gegenbauer transform of order ρ of a function $F(x)$ defined on the interval $-1 \leqslant x \leqslant 1$ as the finite linear integral transformation

$$T\{F(x)\} = \int_{-1}^{1} (1-x^2)^{\rho-\frac{1}{2}} C_n^\rho(x) F(x)\, dx = f^\rho(n) \quad (n = 0, 1, 2,...). \tag{1}$$

The Gegenbauer polynomials $C_n^\rho(x)$ are solutions of the second-order differential equation

$$(1-x^2)y'' - (2\rho+1)xy' + n(n+2\rho)y = 0,$$

which may be written in the self-adjoint form

$$D[(1-x^2)^{\rho+\frac{1}{2}}y'] + n(n+2\rho)(1-x^2)^{\rho-\frac{1}{2}}y = 0. \tag{2}$$

If in (1) we set $x = \cos\theta$, the transform takes the form

$$T\{F(\cos\theta)\} = \int_0^\pi (\sin\theta)^{2\rho} C_n^\rho(\cos\theta) F(\cos\theta)\, d\theta = f^\rho(n). \tag{3}$$

The well-known orthogonality relationship for Gegenbauer polynomials is given by

$$\int_0^\pi (\sin\theta)^{2\rho} C_m^\rho(\cos\theta) C_n^\rho(\cos\theta)\, d\theta = \begin{cases} 0 & (m \neq n), \\ 1/A(\rho,n) & (m = n), \end{cases} \tag{4}$$

where $$A(\rho,n) = 2^{2\rho-1}(\rho+n)n!\,[\Gamma(\rho)]^2/\pi\Gamma(2\rho+n). \tag{5}$$

Under appropriate conditions a function $F(x)$ may be expanded in a series of the Gegenbauer polynomials

$$F(x) = \sum_{n=0}^{\infty} a_n(\rho, n) C_n^{\rho}(x),$$

where

$$a_n = A(\rho, n) \int_{-1}^{1} (1-x^2)^{\rho-\frac{1}{2}} C_n^{\rho}(x) F(x)\, dx = A(\rho, n) f^{\rho}(n).$$

Thus the inversion formula which gives the function $F(x)$ in terms of its transforms is

$$F(x) = \sum_{n=0}^{\infty} A(\rho, n) f^{\rho}(n) C_n^{\rho}(x) = T^{-1}\{f^{\rho}(n)\} \quad (-1 \leqslant x \leqslant 1). \tag{6}$$

The most fundamental property of the Gegenbauer transform is contained in the following theorem:

THEOREM. *Under the transformation T the differential form*

$$R[F(x)] = (1-x^2)F'' - (2\rho+1)xF'$$

is reduced to the algebraic form

$$T\{R[F(x)]\} = -n(n+2\rho) f^{\rho}(n). \tag{7}$$

To prove this we first note that $R[F]$ can be expressed as

$$R[F] = D[(1-x^2)^{\rho+\frac{1}{2}} F'](1-x^2)^{-\rho+\frac{1}{2}}.$$

Integrating the right-hand side by parts twice, we obtain

$$T\{R[F]\} = \int_{-1}^{1} F(x) \left[(1-x^2)^{\rho+\frac{1}{2}} \frac{dC_n^{\rho}}{dx}\right]' dx$$

provided that the function $F(x)$ is sufficiently well-behaved on the interior of the interval $(-1 \leqslant x \leqslant 1)$ and at its end-points. Making use of the differential equation (2) satisfied by $C_n^{\rho}(x)$, we obtain

$$T\{R[F]\} = -n(n+2\rho) \int_{-1}^{1} F(x)(1-x^2)^{\rho-\frac{1}{2}} C_n^{\rho}(x)\, dx$$

$$= -n(n+2\rho) f^{\rho}(n).$$

As an immediate result of this theorem we note that the transform of the iterated differential form $R[R[F]]$ is

$$T\{R^2[F]\} = n^2(n+2\rho)^2 f^{\rho}(n).$$

3. The convolution property

Let $F(x)$ and $G(x)$ denote continuous functions defined on the interval $(-1 \leqslant x \leqslant 1)$. Let

$$T\{F(x)\} = f^{\rho}(n), \qquad T\{G(x)\} = g^{\rho}(n).$$

Then the convolution property determines a function $H(x)$ whose transform is equal to the product of the transforms of $F(x)$ and $G(x)$;

i.e. $$f^{\rho}(n)g^{\rho}(n) = T\{H(x)\} = \int_{-1}^{1} H(x)(1-x^2)^{\rho-\frac{1}{2}}C_n^{\rho}(x)\,dx.$$

We have by definition

$$f^{\rho}(n)g^{\rho}(n)$$

$$= \int_0^{\pi} F(\cos\mu)C_n^{\rho}(\cos\mu)(\sin\mu)^{2\rho}\,d\mu \int_0^{\pi} G(\cos\lambda)C_n^{\rho}(\cos\lambda)(\sin\lambda)^{2\rho}\,d\lambda$$

$$= \int_0^{\pi} F(\cos\mu)(\sin\mu)^{2\rho}\left[\int_0^{\pi} G(\cos\lambda)C_n^{\rho}(\cos\mu)C_n^{\rho}(\cos\lambda)(\sin\lambda)^{2\rho}\,d\lambda\right]d\mu. \quad (8)$$

The addition-formula for Gegenbauer polynomials [(**2**) 177 (20)] states that

$$C_n^{\rho}(\cos\mu)C_n^{\rho}(\cos\lambda) = \frac{\rho+n}{\pi A(\rho,n)}\int_0^{\pi} C_n^{\rho}(\cos\nu)(\sin\alpha)^{2\rho-1}\,d\alpha,$$

where $$\cos\nu = \cos\mu\cos\lambda + \sin\mu\sin\lambda\cos\alpha. \quad (9)$$

Using this addition-formula, we can write the product (8)

$$f^{\rho}(n)g^{\rho}(n) = \frac{\rho+n}{\pi A(\rho,n)}\int_0^{\pi} F(\cos\mu)(\sin\mu)^{2\rho}\times$$

$$\times\left[\int_0^{\pi}\int_0^{\pi} G(\cos\lambda)C_n^{\rho}(\cos\nu)(\sin\lambda)^{2\rho}(\sin\alpha)^{2\rho-1}\,d\alpha d\lambda\right]d\mu. \quad (10)$$

We now introduce a new variable β by the relationship

$$\cos\lambda = \cos\mu\cos\nu + \sin\mu\sin\nu\cos\beta. \quad (11)$$

Then under the transformation of coordinates given by (9) and (11) the element of area $d\alpha d\lambda$ becomes $(\sin\nu/\sin\lambda)\,d\nu d\beta$, where $\sin\nu/\sin\lambda$ is the Jacobian of the transformation. Under this transformation the square region of the (α,λ)-plane given by $(0 \leqslant \alpha \leqslant \pi;\ 0 \leqslant \lambda \leqslant \pi)$ goes into a

square of the same dimensions in the (ν,β)-plane, and hence the iterated integral inside the bracket in (10) becomes

$$\int_0^\pi \int_0^\pi G(\cos\lambda)C_n^\rho(\cos\nu)(\sin\lambda)^{2\rho-1}(\sin\alpha)^{2\rho-1}\sin\nu\, d\nu d\beta, \tag{12}$$

with $\cos\nu$ defined by (9) and $\cos\lambda$ defined by (11). If the integral (12) is substituted into equation (10), and if the order of integration is changed, we obtain the formula

$$f^\rho(n)g^\rho(n) = \frac{\rho+n}{\pi A(\rho,n)}\int_0^\pi C_n^\rho(\cos\nu)(\sin\nu)^{2\rho}H(\cos\nu)\, d\nu$$

$$= \frac{\rho+n}{\pi A(\rho,n)}\, T\{H(\cos\nu)\},$$

where

$$H(\cos\nu) = (\sin\nu)^{1-2\rho}\int_0^\pi \int_0^\pi F(\cos\mu)\times$$

$$\times G(\cos\lambda)(\sin\mu)^{2\rho}(\sin\lambda)^{2\rho-1}(\sin\alpha)^{2\rho-1}\, d\mu d\beta. \tag{13}$$

Thus, if the interchange of operations made above is permissible, we have obtained a function $H(x)$ which is the convolution of the functions $F(x)$ and $G(x)$. This is the convolution property. The formal procedure given above is a direct generalization of that used by Churchill and Dolph in (**1**). For $\rho = \frac{1}{2}$, $C_n^{\frac{1}{2}}(x)$ becomes the Legendre polynomial $P_n(x)$ and the convolution theorem reduces to that given in their paper.

4. An application

I treat here the following generalized equation of one-dimensional heat flow in a non-homogeneous bar†

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial U}{\partial x}\right]-(2\rho+1)x\frac{\partial U}{\partial x} = k\,\frac{\partial U}{\partial t}, \tag{14}$$

where $K = 1-x^2$ is the thermal conductivity of the bar, $(1+2\rho)x\,\partial U/\partial x$ is a continuous source of heat within the solid, k is the diffusivity, and ρ is a constant. The bar is bounded by the planes $x = 1$ and $x = -1$, and its lateral surface is insulated. Initially the temperature is prescribed as

$$U(x,0) = F(x) \quad (-1 < x < 1), \tag{15}$$

† In (**4**) E. J. Scott treats a variation of this same problem using Jacobi transforms, but the solution he gives there is not correct since it fails to satisfy the prescribed boundary conditions.

where $F(x)$ is any function suitably defined on the interval $(-1 < x < 1)$. Because of the nature of the thermal conductivity, the flux of heat must vanish at the ends of the bar, and hence no boundary conditions can be prescribed.

Let $$T\{U(x,t)\} = u^{\rho}(n,t), \qquad T\{F(x)\} = f^{\rho}(n).$$

Applying the transform to equations (14), (15) and making use of property (7), we obtain

$$k\frac{du^{\rho}(n,t)}{dt}+n(n+2\rho)u^{\rho}(n,t) = 0, \qquad u^{\rho}(n,0) = f^{\rho}(n). \tag{16}$$

The solution of the system (16) is

$$u^{\rho}(n,t) = f^{\rho}(n)e^{-n(n+2\rho)t/k}.$$

The inversion theorem then gives the following solution to the heat-conduction problem

$$U(x,t) = \sum_{n=0}^{\infty} A(\rho,n)u^{\rho}(n,t)C_n^{\rho}(x) = \sum_{n=0}^{\infty} A(\rho,n)f^{\rho}(n)e^{-n(n+2\rho)t/k}C_n^{\rho}(x).$$

The problem presented here and its solution are purely formal, but they illustrate the simplicity of Gegenbauer transforms in resolving certain types of boundary-value problems.

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THE SUMMABILITY BY LOGARITHMIC MEANS OF THE DERIVED FOURIER SERIES

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1. LET $f(u)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π, and let

$$f(u) \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nu + b_n \sin nu) = \tfrac{1}{2}a_0 + \sum_1^\infty A_n(u). \tag{1.1}$$

Then the differentiated series of (1.1) at $u = x$ is

$$\sum_1^\infty n(b_n \cos nx - a_n \sin nx) = \sum_1^\infty nB_n(x). \tag{1.2}$$

We write

$$\psi(t) = f(x+t) - f(x-t), \qquad g(t) = \frac{\psi(t)}{4\sin\frac{1}{2}t} - C, \tag{1.3}$$

where C is a function of x.

Let S_n, σ_n, and t_n be respectively the nth partial sum, the first Cesàro mean, and the first logarithmic mean of the series (1.2), so that

$$S_n = \sum_{r=1}^{n} rB_r(x), \tag{1.4}$$

$$\sigma_n = (S_1 + S_2 + \ldots + S_n)/n, \tag{1.5}$$

$$t_n = (S_1 + \tfrac{1}{2}S_2 + \ldots + n^{-1}S_n)/\log n. \tag{1.6}$$

DEFINITION. *The series* (1.2) *is said to be summable* $(R, \log n, 1)$ *to* C *provided that* $t_n \to C$ *as* $n \to \infty$.

The object of this note is to prove the following theorems:

THEOREM 1. *If*

$$\int_t^\pi \frac{|g(u)|}{u}\,du = o\left(\log\frac{1}{t}\right) \quad \text{as } t \to 0, \tag{1.7}$$

then the series (1.2) *is summable* $(R, \log n, 1)$ *to the value* C.

THEOREM 2. *If the condition* (1.7) *holds good, then*

$$\sigma_n = o(\log n).$$

Zygmund (3) has proved that, if

$$g(t) \to 0 \quad \text{as } t \to 0, \tag{1.8}$$

then the series (1.2) is summable $(R, \log n, 1)$ to the value C. Theorem 2 has been proved elsewhere (**2**) under the condition

$$\int_0^t |g(u)|\,du = o(t) \quad \text{as } t \to 0. \tag{1.9}$$

The relation between (1.8) and (1.9) is well known. A relation between (1.7) and (1.9) is contained (**1**) in

LEMMA. *If, as $t \to 0$,*

$$\int_0^t |g(u)|\,du = o(t),$$

then

$$\int_t^\pi \frac{|g(u)|}{u}\,du = o\left(\log\frac{1}{t}\right).$$

On the other hand, if

$$\int_t^\pi \frac{|g(u)|}{u}\,du = o\left(\log\frac{1}{t}\right),$$

then

$$\int_0^t |g(u)|\,du = o\left(t\log\frac{1}{t}\right).$$

In both Theorems 1 and 2 it is enough to consider the special case in which $C = 0$.

To justify this assertion, consider first the case in which

$$f(u) = C\sin(u-x).$$

Then $B_1(x) = C$, $B_n(x) = 0$ $(n \geqslant 2)$, so that the series (1.2) converges to C; *a fortiori*, the conclusions of both theorems are satisfied.

In the general case, write

$$f(u) = C\sin(u-x)+f_1(u).$$

Let $g_1(u)$ be formed from $f_1(u)$ in the same way as $g(u)$ from $f(u)$, but with the C corresponding to $f_1(u)$ taken as 0. Then

$$g_1(t) = g(t)+C\left(1-\frac{\sin t}{2\sin\frac{1}{2}t}\right) = g(t)+o(1).$$

Thus, if $g(u)$ satisfies (1.7), then so does $g_1(u)$, and it is clearly enough to prove both theorems for $g_1(u)$.

2. Proof of Theorem 1

We have
$$rB_r(x) = \frac{1}{\pi}\int\limits_0^\pi \{f(x+t)-f(x-t)\}r\sin rt\,dt$$
$$= \frac{1}{\pi}\int\limits_0^\pi \psi(t)r\sin rt\,dt. \tag{2.1}$$

Hence, by (1.4) and (2.1),
$$S_n = \frac{1}{\pi}\int\limits_0^\pi \psi(t)\,dt\sum_{r=1}^n r\sin rt$$
$$= -\frac{1}{\pi}\int\limits_0^\pi \psi(t)\frac{d}{dt}\Big\{\tfrac{1}{2}+\sum_1^n \cos rt\Big\}\,dt$$
$$= -\frac{1}{\pi}\int\limits_0^\pi \psi(t)\frac{d}{dt}\left\{\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right\}\,dt. \tag{2.2}$$

Now from (1.6) and (2.2) we have
$$t_n = \frac{1}{\log n}\sum_{k=1}^n \frac{S_k}{k}$$
$$= -\frac{1}{\pi\log n}\int\limits_0^\pi \psi(t)\frac{d}{dt}\left\{\frac{1}{2\sin\frac{1}{2}t}\sum_{k=1}^n \frac{\sin(k+\frac{1}{2})t}{k}\right\}\,dt$$
$$= -\frac{1}{2\pi\log n}\int\limits_0^\pi \psi(t)\frac{d}{dt}\left\{\cot\tfrac{1}{2}t\sum_1^n \frac{\sin kt}{k}\right\}\,dt+$$
$$+\frac{1}{2\pi\log n}\int\limits_0^\pi \psi\ t)\frac{d}{dt}\left\{\sum_1^n \frac{\cos kt}{k}\right\}\,dt$$
$$= P+Q, \text{ say.}$$

Consider Q first:
$$|Q| = \frac{1}{2\pi\log n}\left|\int\limits_0^\pi \psi(t)\sum_1^n \sin kt\,dt\right|$$
$$= \frac{1}{2\pi\log n}\left|\int\limits_0^\pi \psi(t)\frac{\cos\frac{1}{2}t-\cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt\right|$$
$$\leqslant \frac{2}{\pi\log n}\int\limits_0^\pi |g(t)|\,dt \to 0 \quad \text{as } n\to\infty.$$

Again

$$P = \frac{1}{4\pi \log n} \int_0^\pi \psi(t) \operatorname{cosec}^2 \tfrac{1}{2}t \sum_1^n \frac{\sin kt}{k}\, dt -$$

$$- \frac{1}{2\pi \log n} \int_0^\pi \psi(t) \cot \tfrac{1}{2}t \sum_1^n \cos kt\, dt$$

$$= P_1 + P_2, \text{ say.}$$

Now

$$P_2 = -\frac{1}{2\pi \log n} \int_0^\pi \psi(t) \cot \tfrac{1}{2}t \frac{\sin(n+\frac{1}{2})t - \sin \frac{1}{2}t}{2 \sin \frac{1}{2}t}\, dt$$

$$= -\frac{1}{\pi \log n} \int_0^\pi g(t) \left\{ \frac{\sin(n+\frac{1}{2})t}{\tan \frac{1}{2}t} - \cos \tfrac{1}{2}t \right\} dt$$

$$= -\frac{1}{\pi \log n} \int_0^\pi g(t) \frac{\sin(n+\frac{1}{2})t}{\tan \frac{1}{2}t}\, dt + \frac{1}{\pi \log n} \int_0^\pi g(t) \cos \tfrac{1}{2}t\, dt$$

$$= J_1 + J_2, \text{ say.} \tag{2.3}$$

Plainly

$$J_2 \to 0 \text{ as } n \to \infty. \tag{2.4}$$

Again

$$J_1 = -\frac{1}{\pi \log n} \int_0^\pi g(t) \frac{\sin(n+\frac{1}{2})t}{\tan \frac{1}{2}t}\, dt$$

$$= -\frac{2}{\pi \log n} \left\{ \int_0^{\pi/n} + \int_{\pi/n}^{\pi} \right\} g(t) \frac{\sin nt}{t}\, dt + o(1)$$

$$= -\frac{2}{\pi \log n} (J_{1,1} + J_{1,2}) + o(1), \text{ say.} \tag{2.5}$$

Now by the lemma

$$|J_{1,1}| \leqslant n \int_0^{\pi/n} |g(t)|\, dt$$

$$= o(\log n) \quad \text{as } n \to \infty. \tag{2.6}$$

Also

$$J_{1,2} = O\left(\int_{\pi/n}^{\pi} \frac{|g(t)|}{t}\, dt \right) = o(\log n). \tag{2.7}$$

Hence, by (2.5), (2.6), and (2.7),

$$J_1 = o(1). \tag{2.8}$$

Therefore, by (2.3), (2.4), and (2.8),

$$P_2 = o(1).$$

Writing $h(n,t)$ for $\sum_{k=1}^{n} \frac{\sin kt}{k}$, we have the following inequalities, which will be needed while considering P_1:

$$|h(n,t)| \leqslant C_1, \quad \text{where } C_1 \text{ is a constant,} \tag{2.9}$$

$$|h(n,t)| \leqslant nt. \tag{2.10}$$

The inequality (2.9) is familiar, and (2.10) is trivial.

Now

$$P_1 = \frac{1}{\pi \log n} \int_0^{\pi} g(t) \operatorname{cosec} \tfrac{1}{2}t\, h(n,t)\, dt$$

$$= \frac{1}{\pi \log n} \int_0^{\pi/n} g(t) \operatorname{cosec} \tfrac{1}{2}t\, h(n,t)\, dt + \frac{1}{\pi \log n} \int_{\pi/n}^{\pi} g(t) \operatorname{cosec} \tfrac{1}{2}t\, h(n,t)\, dt$$

$$= P_{1,1} + P_{1,2}, \text{ say.} \tag{2.11}$$

Using (1.7) and (2.9), we have

$$|P_{1,2}| = O\left\{\frac{1}{\log n} \int_{\pi/n}^{\pi} \frac{|g(t)|}{t}\, dt\right\} = o(1). \tag{2.12}$$

Again, by the lemma and (2.10), we have

$$|P_{1,1}| \leqslant \frac{1}{\pi \log n} \int_0^{\pi/n} |g(t)| \frac{nt}{\sin \frac{1}{2}t}\, dt$$

$$\leqslant \frac{n}{\log n} \int_0^{\pi/n} |g(t)|\, dt$$

$$= o(1). \tag{2.13}$$

Hence, by (2.11), (2.12), and (2.13),

$$P_1 = o(1),$$

and this completes the proof of Theorem 1.

3. Proof of Theorem 2

Easy calculations show that

$$\sigma_n = \frac{2}{\pi n} \int_0^{\pi} g(t) \cos \tfrac{1}{2}t \frac{\sin^2 \frac{1}{2}(n+1)t}{\sin^2 \frac{1}{2}t}\, dt - \frac{2(n+1)}{\pi n} \int_0^{\pi} g(t) \frac{\sin(n+1)t}{2 \sin \frac{1}{2}t}\, dt$$

$$= I_1 + I_2, \text{ say.} \tag{3.1}$$

Using the method employed in estimating J_1, it is easy to see that

$$I_2 = o(\log n). \tag{3.2}$$

Again

$$I_1 = \frac{2}{\pi n}\int\limits_0^{\pi} g(t)\cos\tfrac{1}{2}t\,\frac{\sin^2\frac{1}{2}(n+1)t}{\sin^2\frac{1}{2}t}\,dt$$

$$= \frac{2}{\pi n}\left\{\int\limits_0^{\pi/n} + \int\limits_{\pi/n}^{\pi}\right\} = I_{1,1}+I_{1,2}, \text{ say.} \tag{3.3}$$

Using the lemma we have

$$|I_{1,1}| \leqslant \frac{\pi(n+1)^2}{2n}\int\limits_0^{\pi/n} |g(t)|\,dt$$

$$\leqslant 2\pi n \int\limits_0^{\pi/n} |g(t)|\,dt$$

$$= o(\log n). \tag{3.4}$$

Using (1.7), we find that

$$|I_{1,2}| \leqslant \frac{2\pi}{n}\int\limits_{\pi/n}^{\pi} \frac{|g(t)|}{t^2}\,dt$$

$$\leqslant 2\int\limits_{\pi/n}^{\pi} \frac{|g(t)|}{t}\,dt$$

$$= o(\log n). \tag{3.5}$$

Hence, by (3.3), (3.4), and (3.5), we find that

$$I_1 = o(\log n). \tag{3.6}$$

Therefore, by (3.1), (3.2), and (3.6), we have

$$\sigma_n = o(\log n),$$

which completes the proof of Theorem 2.

Finally we must express our thanks to the referee for some suggestions which improved the presentation and brought in some simplification in the proofs.

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SOME REMARKS ON SCHLICHT FUNCTIONS AND HARMONIC FUNCTIONS OF UNIFORMLY BOUNDED VARIATION

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1. It is familiar that, if $\sigma(z)$ is regular and schlicht in $|z| < 1$, $\sigma(0) = 0$, $\sigma'(0) = 1$, and $\sigma(z)$ maps $|z| < 1$ on to a convex domain, then

$$\operatorname{re}\left\{1+\frac{z\sigma''(z)}{\sigma'(z)}\right\} > 0. \tag{1.1}$$

It follows from this, by a classical result on functions with positive real part, that there exists a real periodic function $U(\psi)$ of bounded variation such that, for $|z| < 1$,

$$\frac{z\sigma''(z)}{\sigma'(z)} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{e^{i\psi}+z}{e^{i\psi}-z}\, dU(\psi),\dagger \tag{1.2}$$

whence, by dividing by z and integrating along the radius from 0 to z, we have

$$\log\sigma'(z) = -\frac{1}{\pi}\int\limits_{-\pi}^{\pi} \log(1-ze^{-i\psi})\, dU(\psi). \tag{1.3}$$

More generally, (1.2) and (1.3) hold (for some periodic U of bounded variation) if σ is regular and schlicht in $|z| < 1$ and

$$\int\limits_{-\pi}^{\pi} \left|\operatorname{re}\left\{\frac{z\sigma''(z)}{\sigma'(z)}\right\}\right| d\theta \leqslant A(\sigma) \quad (z = \rho e^{i\theta},\ \rho < 1).\ddagger \tag{1.4}$$

Conversely, if there is a periodic U of bounded variation such that (1.2) or (1.3) holds, then σ satisfies (1.4).§

† The integrals are Lebesgue or Lebesgue–Stieltjes integrals.

‡ I use $A(b,c,...)$ to denote a positive constant depending only on b, c,..., not necessarily the same on any two occurrences; A by itself will denote a positive absolute constant.

§ In the case in which σ maps $|z| < 1$ on to a convex domain the formula (1.3) is due to Study (**14**). The proof given above [under the more general condition (1.4)] is due essentially to Evans (**2**), 56. See also Paatero (**9**).

Quart. J. Math. Oxford (2), 6 (1955), 59–72.

The formula (1.3) is a limiting form of the well-known Schwarz–Christoffel formula for a function which maps $|z| < 1$ on to the interior of a polygon. Since

$$\mathrm{re}\left\{1+\frac{z\sigma''(z)}{\sigma'(z)}\right\} = \frac{\partial}{\partial\theta}\arg\{iz\sigma'(z)\} \quad (z = \rho e^{i\theta}),$$

the conditions (1.1) and (1.4) may be regarded as concerning the behaviour of the derivative of $\arg\{iz\sigma'(z)\}$ in $|z| < 1$. On the other hand, the conditions under which the classical Schwarz–Christoffel formula is valid (i.e. that σ maps $|z| < 1$ on to the interior of a polygon) may be regarded as concerning the rate of growth of $|\sigma'|$ and the behaviour of $\arg\{iz\sigma'(z)\}$ on the circle $|z| = 1$ itself. For, if σ maps $|z| < 1$ on to the interior of a polygon P, it is continuous in $|z| \leqslant 1$ and regular there save at the points of $|z| = 1$ corresponding to the vertices of P. The angle between the positive direction of the real axis and the tangent to the curve $w = \sigma(e^{i\theta})$ at the point $z = e^{i\theta}$ is here, *qua* function of θ, a step-function; and this angle is $\arg\{iz\sigma'(z)\}$. Further, since P is a rectifiable curve, we have

$$\int_{-\pi}^{\pi} |\sigma'(\rho e^{i\theta})|\, d\theta \leqslant A(\sigma) \quad (\rho < 1).$$

We may therefore ask what are the most general conditions of this latter type under which the formula (1.3) is valid. It is actually more convenient to work here with $\arg\sigma'(z)$ rather than $\arg\{iz\sigma'(z)\}$, and in § 3 I give a set of necessary and sufficient conditions for the validity of (1.3) in terms of the boundary behaviour of $\arg\sigma'(z)$ and the rate of growth of $|\sigma'|$.†

The problem may also be put in a wider setting, when it reduces to finding sets of necessary and sufficient conditions for a function $u(\rho,\theta)$ harmonic in $\rho < 1$ to be the Poisson integral of a function of bounded variation, and these are given in § 2. Most of the results of this section are not essentially new, but they do not seem to have appeared before in a connected form.

The results of this section give also some further properties of the class of schlicht functions σ which satisfy (1.4). This class has been studied in some detail by Paatero (**9**, **10**), who has proved that, if σ is regular and schlicht in $|z| < 1$, satisfies (1.4), and maps $|z| < 1$ on a domain Δ, then

(A) σ is continuous in $|z| \leqslant 1$ save at a finite number of points z_i on $|z| = 1$, while (for each z_i) $|\sigma| \to +\infty$ as $z \to z_i$ in $|z| \leqslant 1$;

† Less general sufficient conditions are given by Komatu (**6**). [I have seen only the review of this paper.]

(B) the boundary Γ of Δ possesses right-hand and left-hand tangents at every finite point, and possesses a proper tangent except at a denumerable infinity of points;

(C) every finite arc of Γ is rectifiable;

(D) for almost all θ, $\{\sigma(z)-\sigma(e^{i\theta})\}/\{z-e^{i\theta}\}$ tends to a finite limit as $z \to e^{i\theta}$ from inside $|z| < 1$, $\sigma(e^{i\theta})$ being defined as the radial limit of $\sigma(z)$ at the point $e^{i\theta}$;

(E) $\sigma''(z)$ possesses for almost all θ a finite limit as $z \to e^{i\theta}$ along any path not touching the circle $|z| = 1$.

The proofs of (B)–(E) given by Paatero are quite short, but that of the key result (A) is very long. In § 4, I use the results of § 2 to give a shorter proof of (A). I give also (in § 5) a new proof of (E), which provides a little additional information, namely that†

(F) $M_\lambda(\rho;\sigma') \leqslant A(\lambda)$ for $0 \leqslant \lambda < \frac{1}{3}$;

(G) $M_\lambda(\rho;\sigma'') \leqslant A(\lambda)$ for $0 \leqslant \lambda < \frac{1}{4}$.

These results are best-possible.

Finally, in § 6, I use an example constructed by Keldysch and Lavrentieff to find the order of the integral on the left of (1.4) for general schlicht σ.

2. In this section we discuss necessary and sufficient conditions for a function $u(\rho,\theta)$ harmonic in $\rho < 1$ to be the Poisson integral of a function of bounded variation. I first list some of the consequences of 'u is the Poisson integral of a U of bounded variation'.

THEOREM 1. *Suppose that $U(\theta)$ is a real function of bounded variation, periodic with period 2π; that $u(\rho,\theta)$ is the Poisson integral of U; that $v(\rho,\theta)$ is that harmonic function conjugate to u which vanishes at the origin; and that $f = u+iv$. Then*

(i) *$u(\rho,\theta)$ is bounded in $\rho < 1$ and for every θ tends to the limit $\frac{1}{2}\{U(\theta+0)+U(\theta-0)\}$ as $\rho \to 1$;*

(ii) *if $\mu = \sup U - \inf U$,*

$$\int_{-\pi}^{\pi} \exp\{\lambda|v(\rho,\theta)|\}\, d\theta \leqslant A(\lambda) \quad \textit{for } \lambda < \pi/\mu;$$

† For any $w(z) = w(\rho e^{i\theta}) = w(\rho,\theta)$ defined in the unit circle I write

$$M_\lambda(w) = M_\lambda(\rho;w) = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |w(\rho,\theta)|^\lambda\, d\theta\right\}^{1/\lambda}.$$

$$\text{(iii)}\ f(z)-f(0) = -\frac{1}{\pi i}\int_{-\pi}^{\pi} \log(1-ze^{-i\psi})\,dU(\psi) \quad (|z|<1);$$

(iv) *$u_\theta(\rho,\theta)$ is the Poisson–Stieltjes integral of U;*

$$\text{(v)}\ \int_{-\pi}^{\pi} |u_\theta(\rho,\theta)|\,d\theta = \int_{-\pi}^{\pi} |du(\rho,\theta)| \leqslant \int_{-\pi}^{\pi} |dU| \quad (\rho<1).$$

Since u is the Poisson integral of U and $v(0)=0$,

$$f(z)-f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\psi}+z}{e^{i\psi}-z}U(\psi)\,d\psi - \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\psi)\,d\psi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{z}{e^{i\psi}-z}U(\psi)\,d\psi$$

$$= \frac{1}{\pi i}\int_{-\pi}^{\pi} U(\psi)d\log(1-ze^{-i\psi})$$

$$= \frac{1}{\pi i}\int_{-\pi}^{\pi} \log(1-ze^{-i\psi})\,dU(\psi),$$

the result of (iii). Differentiating with respect to z and multiplying by iz we have

$$izf'(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{z}{e^{i\psi}-z}\,dU(\psi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\psi}+z}{e^{i\psi}-z}\,dU(\psi),$$

since $\int_{-\pi}^{\pi} dU = 0$, by the periodicity of U. Taking real parts we now obtain (iv);† and (v) follows from (iv) by a simple application of Fubini's theorem. Finally, (i) is familiar,‡ and (ii) is an immediate consequence of a theorem of Zygmund on bounded harmonic functions.§

There is an alternative proof of (ii) as follows. We may suppose that $|U| \leqslant \frac{1}{2}\mu$, when (by a trivial argument)

$$|v(\rho,\theta)| \leqslant \frac{\mu}{\pi}\log\left(\frac{1+\rho}{1-\rho}\right).$$

† This is known; I give the proof for completeness.
‡ See e.g. Zygmund (**15**), 51.
§ See (**15**), 164, Ex. 3.

Now for any absolutely continuous G we have, by the Cauchy–Riemann equations,

$$\rho\frac{d}{d\rho}\int_{-\pi}^{\pi} G(v)\,d\theta = \rho\int_{-\pi}^{\pi} G'(v)v_\rho\,d\theta = -\int_{-\pi}^{\pi} G'(v)u_\theta\,d\theta = -\int_{-\pi}^{\pi} G'(v)\,du$$

[this is practically Prawitz's identity: see Prawitz (**11**)]. If in this we take $G(v) = \exp(\lambda v)$, integrate, and use the above estimate for v, we obtain the desired result.

We note that, if $f = u+iv$ is regular and there exists a periodic U of bounded variation such that the formula (iii) holds for $|z| < 1$, then there exists a constant c such that u is the Poisson integral of $U+c$. The same conclusion holds if u is harmonic and u_θ is the Poisson–Stieltjes integral of a U of bounded variation. To prove these we have only to reverse the relevant arguments above.

Combining this latter remark with a well known theorem on harmonic functions of bounded mean† we have immediately the following converse of Theorem 1 (v).

THEOREM 2.‡ *If $u(\rho,\theta)$ is harmonic in $\rho < 1$ and*

$$\int_{-\pi}^{\pi} |du(\rho,\theta)| = \int_{-\pi}^{\pi} |u_\theta(\rho,\theta)|\,d\theta \leqslant \tau \tag{2.1}$$

when $\rho < 1$, there exists a real periodic function U of total variation not exceeding τ and such that u is the Poisson integral of U.

In other words, in order that u should be the Poisson integral of a function of bounded variation, it is necessary and sufficient that u be of bounded variation in θ, uniformly with respect to ρ.

There is an alternative proof of Theorem 2, more direct than that above. By (2.1), we have

$$-\tau \leqslant u(\rho,\theta) - u(\rho,\theta') \leqslant \tau$$

for all θ and θ'. Integrating these inequalities with respect to θ' over the interval $(-\pi,\pi)$, we obtain

$$-\tau \leqslant u(\rho,\theta) - u(0) \leqslant \tau,$$

i.e. u is bounded.§ Hence u possesses p.p. a finite radial limit U, of which it is the Poisson integral,‖ and, by (2.1), U is of bounded variation when completed in the usual manner.††

† See (**15**), 86. ‡ This is implicit in a result of Evans (**2**), 56.
§ This argument is due to Fichtenholz (**4**). ‖ Fatou (**3**).
†† See Littlewood (**7**), Theorem 20.

We note that for a function u which satisfies (2.1) we have

$$\mu = \sup U - \inf U \leqslant \tfrac{1}{2}\tau,$$

so that the inequality of Theorem 1 (ii) holds for $\lambda < 2\pi/\tau$. In particular, if

$$u_\theta(\rho, \theta) \geqslant -\alpha \tag{2.2}$$

for $\rho < 1$, the inequality of Theorem 1 (ii) holds for $\lambda < 1/2\alpha$, since here u satisfies (2.1) with $\tau = 4\pi\alpha$.

I conclude this section by proving a converse of Theorem 1 (i) and (ii).

THEOREM 3. *If $u(\rho, \theta)$ is harmonic in $\rho < 1$ and tends radially for almost all θ to a periodic function $U(\theta)$ of bounded variation, and if the function v conjugate to u and vanishing at the origin satisfies*

$$M_1(\rho; v) \leqslant A \tag{2.3}$$

when $\rho < 1$, then u is the Poisson integral of U.

Theorem 3 is the analogue for harmonic functions of a known theorem concerning a regular function f which possesses p.p. a radial limit of bounded variation and is such that $M_\lambda(f) = O(1)$ for some $\lambda > 0$.†

Let $f(z) = u+iv$. The condition (2.3) implies first, by a theorem of Kolmogoroff,‡ that $M_\lambda(f) = O(1)$ for $0 < \lambda < 1$, and secondly, that v (and so also f) possesses a radial limit of class L.§ Hence, by a theorem of Smirnoff,‖ $M_1(f) = O(1)$, and so f is the Poisson integral of its boundary values.†† In particular, u is the Poisson integral of U.

3. The theorems of § 2 enable us to give a complete solution to the question raised in § 1 concerning the validity of the formula (1.3). We have in fact

THEOREM 4. *If $\sigma(z)$ is regular and schlicht in $|z| < 1$, $\sigma'(0) = 1$,*

$$\int_{-\pi}^{\pi} \log^+ |\sigma'(\rho e^{i\theta})| \, d\theta \leqslant A, \tag{3.1}$$

and if $\arg \sigma'(\rho e^{i\theta})$, obtained by continuation from the value 0 at $z = 0$, tends p.p. to a periodic function $U(\theta)$ of bounded variation as $\rho \to 1$, then

$$\log \sigma'(z) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \log(1 - ze^{-i\psi}) \, dU(\psi) \tag{3.2}$$

[*and* $\lim\limits_{\rho \to 1} \arg \sigma'(\rho e^{i\theta}) = \tfrac{1}{2}\{U(\theta+0) + U(\theta-0)\}$ *for all* θ].

† The natural analogue of the condition that $M_\lambda(f) = O(1)$ for some $\lambda > 0$ is that $M_\lambda(u) = O(1)$ for some $\lambda > 1$. The condition (2.3), however, is weaker than this, and is, moreover, exactly suited to our application. ‡ See (**15**), 150.

§ See (**15**), 162; that the limit is of class L follows from Fatou's lemma.

‖ See (**15**), 163. †† F. Riesz (**12**).

Conversely, if $\sigma(z)$ is regular and schlicht in $|z| < 1$, $\sigma'(0) = 1$, and there exists a periodic U of bounded variation for which (3.2) *holds, there is a constant c such that*

$$\lim_{\rho\to 1} \arg\sigma'(\rho e^{i\theta}) = \tfrac{1}{2}\{U(\theta+0)+U(\theta-0)\}+c$$

for all θ, and σ satisfies (3.1).

The condition (3.1) is evidently equivalent to

$$\int_{-\pi}^{\pi} |\log|\sigma'||\, d\theta \leqslant A.$$

The first part of the theorem thus follows from Theorems 1 and 3 applied to $f = -i\log\sigma'$, while the converse is an immediate consequence of the remarks at the end of the proof of Theorem 1. The converse shows that the conditions are the best possible of their kind.

4. In this section we consider the class T of functions $\sigma(z)$ regular and schlicht in $|z| < 1$ with $\sigma(0) = 0$, $\sigma'(0) = 1$, and such that

$$\int_{-\pi}^{\pi} |d\arg\sigma'(z)| = \int_{-\pi}^{\pi} \left|\mathrm{re}\left\{\frac{z\sigma''(z)}{\sigma'(z)}\right\}\right| d\theta \leqslant A(\sigma) \quad (z = \rho e^{i\theta},\ \rho < 1). \quad (4.1)$$

The class T contains the functions σ_c which map $|z| < 1$ on to convex domains since for such functions $\mathrm{re}(z\sigma_c''/\sigma_c') > -1$.

I give first a new proof of the theorem of Paatero mentioned in § 1.

THEOREM 5. *If $\sigma(z)$ is of class T, it is continuous in $|z| \leqslant 1$ save at a finite number of points z_i on $|z| = 1$, while (for each z_i) $|\sigma(z)| \to +\infty$ as $z \to z_i$ in $|z| \leqslant 1$.*

The proof depends on a number of lemmas, which for convenience I collect here.

LEMMA 1. *Suppose that $u(\rho,\theta)$ is the Poisson integral of a function $U(\theta)$ of class L, and that $L \leqslant U(\theta) \leqslant M$ when $\alpha \leqslant \theta \leqslant \beta$. Then given η there exists a constant $C = C(\eta)$ such that $L-C(1-\rho) \leqslant u(\rho,\theta) \leqslant M+C(1-\rho)$ whenever $\alpha+\eta \leqslant \theta \leqslant \beta-\eta$ and $0 \leqslant \rho < 1$.*

LEMMA 2. *Suppose that $u(\rho,\theta)$ is the Poisson integral of a function $U(\theta)$ of class L; that v is conjugate to u; that $\theta = \theta_0$ is a point of continuity or of simple discontinuity of U; and that $\Delta U(\theta_0) = U(\theta_0+0)-U(\theta_0-0)$. Then, if $\Delta U(\theta_0) = 0$,*

$$v(\rho,\theta) = o\left\{\log\left(\frac{1}{1-\rho}\right)\right\}$$

as $\rho \to 1$, *while, if* $\Delta U(\theta_0) \neq 0$,

$$v(\rho, \theta) \sim -\frac{\Delta U(\theta_0)}{\pi} \log\left(\frac{1}{1-\rho}\right).$$

LEMMA 3. *Suppose that* Γ *is a simple closed curve, divided at points* a *and* b *into* Γ_1 *and* Γ_2; *that* ϕ *is regular and* $|\phi| \geqslant A > 0$ *in* Δ, *the interior of* Γ; *that* ϕ *is continuous in* $\bar{\Delta}$ *except at* a; *and that* $|\phi(z)| \to +\infty$ *as* $z \to a$ *along* Γ_1, *while* $\phi(z)$ *tends to a limit (possibly infinite) as* $z \to a$ *along* Γ_2. *Then* $|\phi(z)| \to +\infty$ *uniformly as* $z \to a$ *in* $\bar{\Delta}$.

LEMMA 4. *If* $\sigma(z)$ *is regular and schlicht in* $|z| < 1$ *and* $\sigma(0) = 0$, *then*

$$\int_{-\pi}^{\pi} \left|\mathrm{re}\left\{\frac{z\sigma'(z)}{\sigma(z)}\right\}\right| d\theta \leqslant \int_{-\pi}^{\pi} \left|\mathrm{re}\left\{1+\frac{z\sigma''(z)}{\sigma'(z)}\right\}\right| d\theta \quad (z = \rho e^{i\theta},\ \rho < 1).$$

The results of Lemmas 1 and 2 are familiar properties of the Poisson integral, while Lemma 3 is an immediate consequence of a theorem of Lindelöf.† Finally, Lemma 4 is a special case of an inequality of Biernacki [Biernacki (**1**)].

Consider now the proof of the theorem. By Theorem 2 and the hypothesis (4.1), there exists a periodic function $F(\theta)$ of bounded variation such that $\arg \sigma'$‡ is the Poisson integral of F, and (for every θ) $\arg \sigma'(\rho e^{i\theta}) \to F(\theta)$ as $\rho \to 1$. Further, by Lemma 4, we have

$$\int_{-\pi}^{\pi} |d \arg(\sigma/z)| = \int_{-\pi}^{\pi} \left|\frac{\partial}{\partial\theta} \arg(\sigma/z)\right| d\theta = \int_{-\pi}^{\pi} \left|\mathrm{re}\left\{\frac{z\sigma'(z)}{\sigma(z)} - 1\right\}\right| d\theta$$
$$\leqslant \int_{-\pi}^{\pi} \left|\mathrm{re}\left\{\frac{z\sigma'(z)}{\sigma(z)}\right\}\right| d\theta + 2\pi \leqslant A(\sigma).$$

Since σ/z is regular and does not vanish in $|z| < 1$, it follows (again by Theorem 2) that there exists a periodic function $G(\theta)$ of bounded variation such that $\arg(\sigma/z)$§ is the Poisson integral of G, and (for every θ) $\arg\{\sigma(\rho e^{i\theta})/\rho e^{i\theta}\} \to G(\theta)$ as $\rho \to 1$.

I divide the remainder of the proof into a number of stages.

(i) *For every* θ, $\sigma(\rho e^{i\theta})$ *tends to a limit as* $\rho \to 1$ (*infinite values being permitted*).

† See e.g. Littlewood (**7**), Theorem 123. To deduce Lemma 3 from Lindelöf's theorem we have only to apply the latter to $1/\phi$.

‡ Obtained by continuation from the value 0 at $z = 0$.

§ Also obtained by continuation from the value 0 at $z = 0$.

Let R denote the radius $\arg z = \theta$, and suppose that σ does not tend to a limit as $z \to e^{i\theta}$ along R. Then, *if z', z'' are points of R such that $|z'| < |z''|$, the expression* $\arg\{\sigma(z'')-\sigma(z')\}$ *does not tend to a limit as z' and z'' tend to $e^{i\theta}$ along R.* For $\arg\sigma$ tends along R to the limit $\psi = \theta+G(\theta)$. If then λ and Λ are respectively the lower and upper limits of $|\sigma(z)|$ as $z \to e^{i\theta}$ along R (the value $+\infty$ being permitted), we can find a sequence $\{z_n\}$ of points of R with $|z_n|$ increasing to 1 as $n \to \infty$ and such that $\sigma(z_{2n-1}) \to \lambda e^{i\psi}$, while $\sigma(z_{2n}) \to \Lambda e^{i\psi}$. Thus (for large n)

$$\arg\{\sigma(z_{2n})-\sigma(z_{2n-1})\}$$

is near ψ, while $\arg\{\sigma(z_{2n+1})-\sigma(z_{2n})\}$ is near $\psi \pm \pi$, and the statement follows.

We prove that this contradicts the fact that $\arg\sigma'$ tends to the limit $F(\theta)$ along R. By the mean-value theorem in geometrical form, there is at least one point ζ of R between z' and z'' such that the directed tangent to the image of R by σ at the point $\sigma(\zeta)$ is parallel to the directed segment from $\sigma(z')$ to $\sigma(z'')$, i.e. such that

$$\arg\{\sigma(z'')-\sigma(z')\} = \arg\{e^{i\theta}\sigma'(\zeta)\}.$$

Since the expression on the right tends to $\theta+F(\theta)$ as z' and z'' tend to $e^{i\theta}$ along R, we have the necessary contradiction, and (i) is proved.

(ii) *The limit of $\sigma(\rho e^{i\theta})$ as $\rho \to 1$ is finite for all but a finite number of values of θ.*

Applying Lemma 2 to $u = \arg\sigma'$ we see that, when $\Delta F(\theta) \neq 0$,

$$\log|\sigma'(\rho e^{i\theta})| \sim \frac{\Delta F(\theta)}{\pi}\log\left(\frac{1}{1-\rho}\right), \qquad |\sigma'(\rho e^{i\theta})| = (1-\rho)^{-\Delta F(\theta)\{1+o(1)\}/\pi} \tag{4.2}$$

as $\rho \to 1$. Hence, if $\Delta F(\theta) < \pi$, there is a positive $\delta = \delta(\theta)$ such that $|\sigma'(\rho e^{i\theta})| \leqslant (1-\rho)^{-(1-\delta)}$ for ρ near 1, whence also

$$|\sigma(\rho e^{i\theta})| \leqslant \int_0^\rho |\sigma'(re^{i\theta})|\, dr \leqslant A(\sigma) + \int_0^\rho (1-r)^{-(1-\delta)}\, dr < A(\sigma, \theta)$$

Since F is of bounded variation, $\Delta F(\theta) < \pi$ for all but a finite number of values of θ, and the result follows.†

We now define $\sigma(e^{i\theta}) = \lim_{\rho \to 1} \sigma(\rho e^{i\theta})$, the value $\infty e^{i\psi}$ being permitted. We prove next

(iii) *For every θ_0, $\sigma(e^{i\theta})$ tends to a limit as $\theta \to \theta_0+0$, and also as $\theta \to \theta_0-0$ (infinite values being permitted).*

Suppose that $\sigma(e^{i\theta})$ does not tend to a limit as $\theta \to \theta_0+0$. Since

† From (4.2) and the fact that $\arg\sigma'$ possesses a finite radial limit, it follows also that $|\sigma(\rho e^{i\theta})| \to +\infty$ as $\rho \to 1$ whenever $\Delta F(\theta) > \pi$.

$\arg\sigma(e^{i\theta})$ tends to the (finite) limit $\theta_0+G(\theta_0+0)$ as $\theta\to\theta_0+0$, it follows, exactly as in the proof of (i), that $\arg\{\sigma(e^{i\theta''})-\sigma(e^{i\theta'})\}$ does not tend to a limit as θ' and θ'' tend to θ_0 in such a manner that $\theta_0<\theta'<\theta''$.

The remainder of the argument, while similar in general principle to that of (i), requires a little extra care, since σ' is not defined on $|z|=1$. Let $l=F(\theta_0+0)$. Then, given ϵ, there exists $\delta=\delta(\epsilon)$ such that $l-\epsilon\leqslant F(\theta)\leqslant l+\epsilon$ whenever $\theta_0<\theta<\theta_0+\delta$. Suppose now that $\theta_0<\theta'<\theta''<\theta_0+\delta$. By the mean-value theorem in geometrical form, there is at least one point ζ on the arc $\theta'\leqslant\theta\leqslant\theta''$ of the circle $|z|=\rho$ such that the directed tangent to the image of this arc by σ at the point $\sigma(\zeta)$ is parallel to the directed segment from $\sigma(\rho e^{i\theta'})$ to $\sigma(\rho e^{i\theta''})$, i.e. such that
$$\arg\{\sigma(\rho e^{i\theta''})-\sigma(\rho e^{i\theta'})\}=\arg\{i\zeta\sigma'(\zeta)\}.$$
Now, by Lemma 1 applied to $u=\arg\sigma'$, we have
$$l-\epsilon+o(1)\leqslant\arg\sigma'(\zeta)\leqslant l+\epsilon+o(1),$$
uniformly as $\zeta=\rho e^{i\theta}$ tends in the sector $\theta'\leqslant\theta\leqslant\theta''$ to the circumference of the unit circle. We have therefore
$$l+\tfrac{1}{2}\pi+\theta_0-\epsilon\leqslant\arg\{\sigma(e^{i\theta''})-\sigma(e^{i\theta'})\}\leqslant l+\tfrac{1}{2}\pi+\theta_0+\epsilon,$$
and this is impossible since $\arg\{\sigma(e^{i\theta''})-\sigma(e^{i\theta'})\}$ does not tend to a limit as θ'' and θ' tend to θ_0.

(iv) *σ is continuous (in $|z|\leqslant 1$) at every point $e^{i\theta_0}$ such that $\sigma(e^{i\theta_0})$ is finite.*

The function $\phi(z)=\{\sigma(z^3)\}^{\frac{1}{3}}$ is regular and schlicht in $|z|<1$, and
$$\int_{-\pi}^{\pi}|\phi(\rho e^{i\theta})|\,d\theta=\int_{-\pi}^{\pi}|\sigma(\rho^3e^{3i\theta})|^{\frac{1}{3}}\,d\theta=\int_{-\pi}^{\pi}|\sigma(\rho^3e^{i\psi})|^{\frac{1}{3}}\,d\psi<A(\sigma).\dagger \quad (4.3)$$
Since the boundary values of σ exist everywhere and possess left-hand and right-hand limits (possibly infinite) at each point of $|z|=1$, the same is true also of ϕ, and further, by (4.3), ϕ is the Poisson integral of its boundary values.‡ Let $\phi(e^{i\theta})=\lim_{\rho\to1}\phi(\rho e^{i\theta})$, and write $\theta_1=\frac{1}{3}\theta_0$. If $\sigma(e^{i\theta_0})$ is finite, so also is $\phi(e^{i\theta_1})$. It follows from Lemma 2 that neither $\operatorname{re}\phi(e^{i\theta})$ nor $\operatorname{im}\phi(e^{i\theta})$ can have a simple discontinuity at $\theta=\theta_1$, for, if one does, $\phi(\rho e^{i\theta_1})$ is unbounded and $\phi(e^{i\theta_1})$ is not finite. Nor can either the left-hand or the right-hand limit of $\phi(e^{i\theta})$ at $\theta=\theta_1$ be infinite. For suppose that $|\phi(e^{i\theta})|\to+\infty$ as $\theta\to\theta_1+0$ (say). Then $|\phi(z)|\to+\infty$ as $z\to e^{i\theta_1}$ along a path Γ_1 in $|z|<1$ meeting $|z|=1$ tangentially from above.§ Also, since ϕ is schlicht and $\phi(0)=0$, $|\phi|\geqslant A>0$ in the

† See e.g. Littlewood (**7**), Theorem 249.

‡ F. Riesz (**12**).

§ Fatou (**3**).

neighbourhood of $z = e^{i\theta_1}$. Hence, by Lemma 3 (applied to the curve consisting of Γ_1, part of the radius $\arg z = \theta_1$ near $z = e^{i\theta_1}$, and some curve in $|z| < 1$ joining these two), $|\phi(z)| \to +\infty$ as $z \to e^{i\theta_1}$ along the radius, and so again $\phi(e^{i\theta_1})$ is not finite. Thus $\phi(e^{i\theta})$ is continuous at $\theta = \theta_1$, and the result follows.

Finally, we have

(v) *If $\sigma(e^{i\theta_0})$ is infinite, then $|\sigma(z)| \to +\infty$ uniformly as $z \to e^{i\theta_0}$ in $|z| \leqslant 1$.*

This is now an immediate consequence of (iii) and Lemma 3 [for, by (iv), $\phi(z)$ is continuous in a neighbourhood of $z = e^{i\theta_0}$ (in $|z| \leqslant 1$) except at $e^{i\theta_0}$ itself].

5. We turn now to the derivatives of functions of the class T. It follows from Theorems 1 and 2 applied to $u = \arg \sigma'$ that, if σ belongs to T, there exists a positive λ_0 such that

$$M_\lambda(\sigma') \leqslant A(\lambda) \tag{5.1}$$

for $\lambda < \lambda_0$. In particular, if σ is a σ_c (i.e. maps $|z| < 1$ on a convex domain), $u = \arg \sigma'$ satisfies (2.2) with $\alpha = 1$, so that (5.1) holds for $0 \leqslant \lambda < \frac{1}{2}$. For general σ of T, however, the inequality (5.1) holds only for $0 \leqslant \lambda < \frac{1}{3}$; this follows immediately from Prawitz's identity [Prawitz (**11**)] applied to σ' combined with the inequality

$$|\sigma'(\rho e^{i\theta})| \leqslant (1+\rho)/(1-\rho)^3.$$

Thus for the functions of class T the similarity between a schlicht σ and the function $z/(1-z)^2$ extends to the λth mean of σ' for $\lambda < 1$. We may show further (again for the functions of this class) that this similarity extends also to the means of σ''. In particular, we have

THEOREM 6. *If $\sigma(z)$ belongs to the class T, then* (i) $M_\lambda(\sigma') \leqslant A(\lambda)$ *for* $0 \leqslant \lambda < \frac{1}{3}$; (ii) $M_\lambda(\sigma'') \leqslant A(\lambda)$ *for* $0 \leqslant \lambda < \frac{1}{4}$.

After the preceding remarks it remains only to prove (ii). If $s > \lambda > 0$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma''|^\lambda \, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma'|^\lambda \left|\frac{\sigma''}{\sigma'}\right|^\lambda d\theta \leqslant \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma'|^s \, d\theta\right\}^{\lambda/s} \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sigma''}{\sigma'}\right|^{\lambda s/(s-\lambda)} d\theta\right\}^{(s-\lambda)/s},$$

by Hölder's inequality. In this take

$$\lambda = \tfrac{1}{4}-\delta < \tfrac{1}{4}, \qquad s = (\tfrac{1}{4}-\tfrac{1}{2}\delta)/(\tfrac{3}{4}+\tfrac{1}{2}\delta).$$

Then $s < \frac{1}{3}$ and $\lambda s/(s-\lambda) < 1$. Since $M_s(\sigma') = O(1)$ for $s < \frac{1}{3}$ [by (i)] and $M_t(\sigma''/\sigma') = O(1)$ for $t < 1$ [by Kolmogoroff's theorem,† applied to $z\sigma''/\sigma'$], (ii) now follows.

Combining (ii) with a known theorem concerning regular functions of bounded mean‡ we have immediately the corollary:

COROLLARY.§ *If $\sigma(z)$ belongs to the class T, then, for almost all θ, σ'' tends to a finite limit as $z \to e^{i\theta}$ along any path not touching the circle $|z| = 1$.*

A proof similar to that of Theorem 6 (ii), using the known result that $M_s(\sigma) = O(1)$ for $s < \frac{1}{2}$ in place of Theorem 6 (i), shows that the result of Theorem 6 (i) continues to hold if we replace (4.1) by the weaker‖ condition

$$\int_{-\pi}^{\pi} |d \arg \sigma| = \int_{-\pi}^{\pi} \left|\mathrm{re}\left(\frac{z\sigma'}{\sigma}\right)\right| d\theta \leqslant A(\sigma). \tag{5.2}$$

In particular, if σ_* maps $|z| < 1$ on to a star-shaped domain, σ_* satisfies (5.2) and we have $M_\lambda(\sigma_*') \leqslant A(\lambda)$ for $0 \leqslant \lambda < \frac{1}{3}$ [and, for almost all θ, σ_*' tends to a finite limit as $z \to e^{i\theta}$ along any path not touching the circle $|z| = 1$].†† It follows also that, if σ_c maps $|z| < 1$ on to a convex domain, then $M_\lambda(\sigma_c'') \leqslant A(\lambda)$ for $0 \leqslant \lambda < \frac{1}{3}$ [since $z\sigma_c'$ is a σ_*].

The example $\sigma(z) = z/(1-z)^2$ shows that the results of Theorem 6 (i) [under either (4.1) or (5.2)] and (ii) are best-possible. There is thus a marked difference between the behaviour of functions σ of class T in general and that of the functions σ_c.

6. The condition (4.1) is an extremely restrictive one. For general schlicht σ we have only

$$\int_{-\pi}^{\pi} |d \arg \sigma'| = \int_{-\pi}^{\pi} \left|\mathrm{re}\left(\frac{z\sigma''}{\sigma'}\right)\right| d\theta = O\{(1-\rho)^{-1}\}, \tag{6.1}$$

and this is best-possible in that the $(1-\rho)^{-1}$ on the right cannot be replaced by $(1-\rho)^{-\alpha}$ for any $\alpha < 1$, even in the case in which σ maps $|z| < 1$ on to a domain bounded by a rectifiable curve.

The direct result is an immediate consequence of the known result that $|\sigma''/\sigma'| = O\{(1-\rho)^{-1}\}$. To prove (6.1) best-possible we require the following theorem, which seems of independent interest.

† See (**15**), 150. ‡ See (**15**), 162. § Paatero (**10**).
‖ See Lemma 4 of § 4. †† Lusin and Privaloff (**8**).

THEOREM 7. *If $\sigma(z)$ is regular and schlicht in $|z| < 1$, and, as $\rho \to 1$,*

$$\int_{-\pi}^{\pi} |d \arg \sigma'| = O\{(1-\rho)^{-\alpha}\},$$

where $\alpha < 1$, then

$$\int_{-\pi}^{\pi} |\log \sigma'| \, d\theta = O(1) \tag{6.2}$$

[*and, for almost all θ, σ' possesses a finite limit as $z \to e^{i\theta}$ along any path not touching $|z| = 1$*].

Writing $\sigma'(\rho e^{i\theta}) = Re^{i\Phi}$ we have, by Prawitz's identity,

$$\rho \frac{d}{d\rho} \int_{-\pi}^{\pi} \log^2 |\sigma'| \, d\theta = 2 \int_{-\pi}^{\pi} \log R \, d\Phi \leqslant 2 \sup_{|z|=\rho} \log |\sigma'| \int_{-\pi}^{\pi} |d\Phi|$$
$$\leqslant A(1-\rho)^{-\alpha} \log 1/(1-\rho),$$

so that

$$\int_{-\pi}^{\pi} \log^2 |\sigma'| \, d\theta = O(1).$$

Hence, by M. Riesz's theorem on conjugate functions (the simple case of index 2),

$$\int_{-\pi}^{\pi} |\log \sigma'|^2 \, d\theta = O(1),$$

and (6.2) now follows by Schwarz's inequality.

If $\alpha = 1$, this argument gives only

$$\int_{-\pi}^{\pi} |\log \sigma'| \, d\theta = O\{\log 1/(1-\rho)\}.$$

This, however, may be improved† to

$$\int_{-\pi}^{\pi} |\log \sigma'| \, d\theta = O\{\sqrt{\log 1/(1-\rho)}\}.$$

For, by Spencer's identity,‡ we have

$$\rho \frac{d}{d\rho} \int_{-\pi}^{\pi} \log^2 |\sigma'(\rho e^{i\theta})| \, d\theta = 2 \int_0^{\rho} \int_{-\pi}^{\pi} \left| \frac{\sigma''(re^{i\phi})}{\sigma'(re^{i\phi})} \right|^2 r \, d\phi dr \leqslant A \int_0^{\rho} \frac{dr}{(1-r)^2} \leqslant \frac{A}{1-\rho},$$

and the proof may now be completed by an argument similar to that of Theorem 7.

† It does not seem to be known whether $\int_{-\pi}^{\pi} \log^+ |\sigma'| \, d\theta = O(1)$ for all schlicht σ, though it seems very unlikely. A similar remark applies also to the two results of Theorem 6.

‡ Spencer (**13**), equation (1.5).

Return now to (6.1). Keldysch and Lavrentieff (**5**) have given an example of a schlicht σ which maps $|z| < 1$ on to a domain bounded by a rectifiable curve and is such that $\log|\sigma'|$ is not the Poisson integral of its boundary values. For such a σ we must therefore have

$$\int_{-\pi}^{\pi} |\log \sigma'|\, d\theta \neq O(1),$$

and so (again for this σ)

$$\int_{-\pi}^{\pi} |d \arg \sigma'| = O\{(1-\rho)^{-\alpha}\}$$

is false for every $\alpha < 1$. This proves the statement.

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THE CLASSICAL CANONICAL FORM OF A NILPOTENT MATRIX

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LET A be a nilpotent matrix of order n and index r, so that

$$A^r = 0, \qquad A^{r-1} \neq 0. \tag{1}$$

Let X_1 be a *non-zero column* of A^{r-1} and $X_2, X_3,..., X_r$ the *corresponding* columns of $A^{r-2}, A^{r-3},..., A$ and A^0 $(= I)$ respectively. Then we have

$$AX_1 = 0, \quad AX_2 = X_1, \quad AX_3 = X_2, \quad ..., \quad AX_r = X_{r-1}. \tag{2}$$

From (2) we deduce that the r vectors $X_1, X_2,..., X_r$ are linearly independent.

Let T be any *non-singular* matrix of order n with $X_1, X_2,..., X_r$ as its first r columns. Then in virtue of (2) we immediately have

$$T^{-1}AT = \begin{pmatrix} J_r & C \\ O & B \end{pmatrix}, \tag{3}$$

where J_r is the $r \times r$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 0 & 1 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 \end{pmatrix}$$

and C is an $r \times (n-r)$ submatrix, B is an $(n-r)\times(n-r)$ submatrix, O is an $(n-r)\times r$ null block. If $r = n$, then we have $T^{-1}AT = J_n$. Thus every nilpotent matrix of order n and index n is similar to J_n, and this is the classical canonical form of A in this case.

If $r < n$, we proceed as follows. Since A is nilpotent, it follows from (3) that B is a nilpotent matrix of order $n-r$. *Assume* that every nilpotent matrix of order m $(< n)$ can be transformed into the classical canonical form

$$\operatorname{diag}(J_{\rho_1}, J_{\rho_2},..., J_{\rho_k}) \quad (\rho_1+\rho_2+...+\rho_k = m),$$

where J_r is defined as above. Then B is similar to $\operatorname{diag}(J_{r_1}, J_{r_2},..., J_{r_m})$

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$(r_1+r_2+...+r_m = n-r)$, and therefore, by (3), A can be transformed by means of a non-singular matrix P into the form

$$P^{-1}AP = \begin{pmatrix} J_r & D_{rr_1} & D_{rr_2} & . & . & . & D_{rr_m} \\ 0 & J_{r_1} & 0 & . & . & . & 0 \\ 0 & 0 & J_{r_2} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & J_{r_m} \end{pmatrix}, \tag{4}$$

where D_{rs} is an $r\times s$ block $(s = r_1, r_2,..., r_m)$.

We now proceed to annihilate these unwanted blocks D_{rs}. First observe that, since $A^r = 0$, $(P^{-1}AP)^r = 0$, and therefore from (4), taking the rth power, we get

$$J_r^r = 0 \quad \text{(which clearly holds)},$$

$$J_s^r = 0 \quad \text{(which implies that } s \leqslant r \text{ for } s = r_1, r_2,..., r_m),$$

and $$J_r^{r-1} D_{rs}+J_r^{r-2} D_{rs} J_s+...+J_r D_{rs} J_s^{r-2}+D_{rs} J_s^{r-1} = 0$$

for $$s = r_1, r_2,..., r_m.$$

But $s \leqslant r$ and $J_s^s = 0$. Hence the above relation is really equivalent to

$$J_r^{r-1} D_{rs}+J_r^{r-2} D_{rs} J_s+...+J_r^{r-s} D_{rs} J_s^{s-1} = 0. \tag{5}$$

Thus D_{rs} is an $r\times s$ block $(s \leqslant r)$ satisfying the relation (5).

Now let
$$Q = \begin{pmatrix} I_r & G_{rr_1} & G_{rr_2} & . & . & . & G_{rr_m} \\ 0 & I_{r_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & I_{r_m} \end{pmatrix},$$

where I_k is the unit matrix of order k, and G_{rs} is an $r\times s$ block *undetermined as yet*. Then clearly

$$Q^{-1} = \begin{pmatrix} I_r & -G_{rr_1} & -G_{rr_2} & . & . & . & -G_{rr_m} \\ 0 & I_{r_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & I_{r_m} \end{pmatrix},$$

and therefore

$$Q^{-1}P^{-1}APQ = \begin{pmatrix} J_r & H_{rr_1} & H_{rr_2} & . & . & . & H_{rr_m} \\ 0 & J_{r_1} & 0 & . & . & . & 0 \\ 0 & 0 & J_{r_2} & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & J_{r_m} \end{pmatrix}, \tag{6}$$

where $$H_{rs} = J_r G_{rs}+D_{rs}-G_{rs} J_s \quad (s = r_1, r_2,..., r_m). \tag{7}$$

We now ask this question:

Is it possible to find G_{rs} such that $H_{rs} = 0$?

In other words the problem is this. D_{rs} is an $r \times s$ matrix ($s \leqslant r$) satisfying the relation (5). Is it possible to find an $r \times s$ matrix G_{rs} such that

$$D_{rs} = G_{rs} J_s - J_r G_{rs}? \tag{8}$$

I show that the answer to this question is in the affirmative. To see what (5) and (8) imply, let

$$D_{rs} = \begin{pmatrix} d_{11} & d_{12} & . & . & . & d_{1s} \\ d_{21} & d_{22} & . & . & . & d_{2s} \\ . & . & . & . & . & . \\ d_{r1} & d_{r2} & . & . & . & d_{rs} \end{pmatrix}, \quad G_{rs} = \begin{pmatrix} g_{11} & g_{12} & . & . & . & g_{1s} \\ g_{21} & g_{22} & . & . & . & g_{2s} \\ . & . & . & . & . & . \\ g_{r1} & g_{r2} & . & . & . & g_{rs} \end{pmatrix}.$$

Simple calculations show that (5) implies

$$\left.\begin{array}{l} d_{r1} = 0 \\ d_{r2}+d_{r-1,1} = 0 \\ d_{r3}+d_{r-1,2}+d_{r-2,1} = 0 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ d_{rs}+d_{r-1,s-1}+d_{r-2,s-2}+\ldots+d_{r-s+1,1} = 0 \end{array}\right\}. \tag{9}$$

These are the relations satisfied by the elements d_{ij} of D_{rs}, while (8) implies

$$\left.\begin{array}{lllll} -g_{21} = d_{11}, & g_{11}-g_{22} = d_{12}, & g_{12}-g_{23} = d_{13}, & \ldots & g_{1,s-1}-g_{2s} = d_{1s} \\ -g_{31} = d_{21}, & g_{21}-g_{32} = d_{22}, & g_{22}-g_{33} = d_{23}, & \ldots & g_{2,s-1}-g_{3s} = d_{2s} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -g_{r1} = d_{r-1,1}, & g_{r-1,1}-g_{r2} = d_{r-1,2}, & g_{r-1,2}-g_{r3} = d_{r-1,3}, & \ldots & g_{r-1,s-1}-g_{rs} = d_{r-1,s} \\ 0 = d_{r1}, & g_{r1} = d_{r2}, & g_{r2} = d_{r3}, & \ldots & g_{r,s-1} = d_{rs} \end{array}\right\}. \tag{10}$$

Now, since (9) holds, it is clear that the equations (10) are *consistent*. Moreover, from (10) it follows that $g_{11}, g_{12},\ldots, g_{1s}$ can be arbitrarily chosen and then the rest of the g_{ij} can be uniquely determined in terms of these and the various d_{ij}. Thus a matrix G_{rs} can be found such that (8) holds. The answer to the question raised above is therefore in the affirmative. It is possible to find G_{rs} such that $H_{rs} = 0$ and then (6) becomes

$$Q^{-1}P^{-1}APQ = \operatorname{diag}(J_r, J_{r_1}, J_{r_2},\ldots, J_{r_m}). \tag{11}$$

Thus on the assumption that every nilpotent matrix of order less than n can be transformed into the classical canonical form, we have proved that the same holds for any nilpotent matrix of order n.

The case $n = 1$ is trivial and the case $n = 2$ is easy to work out. Hence by induction every nilpotent matrix of order n can be transformed into the classical canonical form

$$\operatorname{diag}(J_r, J_{r_1}, J_{r_2},\ldots, J_{r_m}) \quad (r+r_1+r_2+\ldots+r_m = n).$$

Note. It is quite simple to determine the indices $r, r_1, r_2,..., r_m$ in terms of the ranks of A and its various powers.

Suppose that the classical canonical form of A is

$$\operatorname{diag}(J_1, J_1,..., J_1, J_2, J_2,..., J_2,..., J_r, J_r,..., J_r), \tag{i}$$

in which J_i occurs μ_i times, $\mu_1, \mu_2,..., \mu_r$ being non-negative integers. Suppose that the ranks of A, A^2, A^3,..., A^{r-1} are respectively

$$\rho_1, \rho_2,..., \rho_{r-1} \quad (A^r = 0).$$

Now observe that the rank of J_s is $s-1$, that of J_s^2 is $s-2$,..., and that of J_s^k is $s-k$ ($k \leqslant s$). Hence, since similar matrices have the same rank, we get by (i) and its various powers

$$\left.\begin{array}{ll} \mu_2+2\mu_3+...+(r-1)\mu_r &= \rho_1 \\ \mu_3+2\mu_4+...+(r-2)\mu_r &= \rho_2 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot \\ \mu_{r-1}+2\mu_r &= \rho_{r-2} \\ \mu_r &= \rho_{r-1} \end{array}\right\}. \tag{ii}$$

Also, since (i) is of order n, we get

$$\mu_1+2\mu_2+3\mu_3+...+r\mu_r = n. \tag{iii}$$

We express these equations in the form

$$\begin{pmatrix} 1 & 2 & 3 & . & . & . & r \\ 0 & 1 & 2 & . & . & . & r-1 \\ 0 & 0 & 1 & . & . & . & r-2 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ \mu_r \end{pmatrix} = \begin{pmatrix} n \\ \rho_1 \\ \rho_2 \\ . \\ \rho_{r-1} \end{pmatrix},$$

whence, by solving,

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ . \\ . \\ . \\ . \\ \mu_r \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n \\ \rho_1 \\ \rho_2 \\ . \\ . \\ . \\ \rho_{r-1} \end{pmatrix}.$$

I am grateful to Dr. W. L. Ferrar for the interest he took in the preparation of this paper.

ON THE INTEGRABILITY OF FUNCTIONS DEFINED BY TRIGONOMETRIC SERIES (II)

By P. HEYWOOD (*Edinburgh*)

[Received 3 September 1954]

1. R. P. Boas (**1**) has proved that, *if λ_n is decreasing for large values of n, if*

$$\tfrac{1}{2}\lambda_0+\sum_{1}^{\infty}\lambda_n = 0, \tag{1.1}$$

and if

$$f(x) = \tfrac{1}{2}\lambda_0+\sum_{1}^{\infty}\lambda_n \cos nx, \tag{1.2}$$

then $f(x)/x$ is $L(0,\pi)$ if and only if $\sum \lambda_n \log n$ converges. This theorem arose as a limiting case of a theorem on the integrability of $f(x)/x^{\gamma}$ for $0<\gamma<1$. In a previous note (**2**) I considered the integrability of $f(x)/x^{\gamma}$ for values of γ greater than 1, but I overlooked the fact that a variant of a method used in (**2**) gives a simpler proof of the above result and also shows that it is not necessary to assume that λ_n is ultimately monotonic. In the present note the following theorem will be proved.

Theorem. *If λ_n is ultimately positive, if* (1.1) *holds, and if $f(x)$ is defined by* (1.2),† *then $f(x)/x$ is $L(0,\pi)$ if and only if $\sum \lambda_n \log n$ converges.*

2. To prove this we observe, as in (**2**), that (1.1) implies that

$$f(x) = -\sum_{1}^{\infty}\lambda_n(1-\cos nx). \tag{2.1}$$

Suppose that λ_n is positive for all values of n greater than N, and let

$$g(x) = \sum_{N+1}^{\infty}\lambda_n(1-\cos nx).$$

Then, by (2.1),

$$f(x)+g(x) = -\sum_{1}^{N}\lambda_n(1-\cos nx).$$

Since each of the N functions $(1-\cos nx)/x$ $(n = 1, 2,..., N)$ is $L(0,\pi)$, it follows that $f(x)/x$ is $L(0,\pi)$ if and only if $g(x)/x$ is $L(0,\pi)$.

I shall first show that, if $\sum \lambda_n \log n$ converges, then $g(x)/x$ is $L(0,\pi)$. In view of the preceding remarks, this will prove one half of the theorem. Since $g(x)$ is continuous and positive for all values of x, it will be sufficient

† We see that the series which defines $f(x)$ is uniformly convergent for all values of x if we compare it with the absolutely convergent series on the left of (1.1).

Quart. J. Math. Oxford (2), 6 (1955), 77–79.

to show that

$$I_\delta \equiv \int_\delta^\pi \frac{g(x)}{x}\,dx = \int_\delta^\pi \sum_{N+1}^\infty \frac{\lambda_n(1-\cos nx)}{x}\,dx \tag{2.2}$$

is bounded for all small positive values of δ.

If δ is fixed and positive, the series on the right of (2.2) is uniformly convergent for $\delta \leqslant x \leqslant \pi$. We may therefore integrate term by term, and we find that

$$I_\delta = \sum_{N+1}^\infty \lambda_n \int_\delta^\pi \frac{1-\cos nx}{x}\,dx. \tag{2.3}$$

Let $$J_n = \int_0^\pi \frac{1-\cos nx}{x}\,dx = \int_0^{n\pi} \frac{1-\cos y}{y}\,dy.$$

Then, as n tends to infinity,

$$J_n \sim \int_1^{n\pi} \frac{dy}{y} \sim \log n. \tag{2.4}$$

We are supposing that $\sum \lambda_n \log n$ is convergent. Hence, since λ_n is ultimately positive, it follows from (2.4) that $\sum \lambda_n J_n$ is convergent. But (2.3) shows that

$$I_\delta < \sum_{N+1}^\infty \lambda_n J_n$$

for any small positive value of δ. Hence I_δ is bounded for all small positive values of δ, and we have proved one half of the theorem.

Conversely, suppose that $f(x)/x$ is $L(0,\pi)$, so that $g(x)/x$ is $L(0,\pi)$. For any positive integer p we have

$$\begin{aligned}\sum_{N+1}^{N+p} \lambda_n J_n &= \sum_{N+1}^{N+p} \lambda_n \int_0^\pi \frac{1-\cos nx}{x}\,dx \\ &= \int_0^\pi \sum_{N+1}^{N+p} \frac{\lambda_n(1-\cos nx)}{x}\,dx \\ &\leqslant \int_0^\pi \frac{g(x)}{x}\,dx.\end{aligned}$$

Hence the series of positive terms

$$\sum_{N+1}^\infty \lambda_n J_n$$

is convergent. It follows from (2.4) that $\sum \lambda_n \log n$ converges. This completes the proof of the theorem.

I wish to express my thanks to the referee, who suggested a change in the style of the second section of this note.

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COMPOSITE MATRICES

By J. G. MAULDON (*Oxford*)

[Received 16 September 1954]

LET $\mathfrak{R}$ be a commutative ring, let $\mathfrak{S}$ be a ring of commutative square matrices of order n with elements in $\mathfrak{R}$, and let $\mathbf{A}$ be a square matrix of order m with elements in $\mathfrak{S}$. Then $\mathbf{A}$ can be regarded as a composite matrix over $\mathfrak{R}$, and its *ground matrix* $\mathbf{a}$ is the simple square matrix of order mn from which $\mathbf{A}$ can be derived by partitioning.

Clearly, taking determinants, we have

$$|\mathbf{A}| = \Delta \in \mathfrak{S}, \qquad \|\mathbf{A}\| = |\Delta| \in \mathfrak{R}, \qquad |\mathbf{a}| \in \mathfrak{R}.$$

In a recent paper [(**1**), Theorem V] Afriat has proved, among other results, that, if $\mathfrak{R}$ is an algebraically closed field, then

$$|\mathbf{a}| = \|\mathbf{A}\|, \tag{1}$$

and it is the purpose of this note to extend the validity of (1) to the case when $\mathfrak{R}$ is any commutative ring.

Afriat's own proof of Theorem V is valid for any commutative ring as far as the equation

$$|\mathbf{b}_1|\,\|\mathbf{A}\| = |\mathbf{b}_1|\,|\mathbf{a}|. \tag{2}$$

Now let $\mathfrak{R}^0$ be obtained from $\mathfrak{R}$ by adjoining a generator λ, subject to the relations

$$\lambda r = r\lambda \quad (r \in \mathfrak{R}).$$

Let $\mathbf{a}^0$ be obtained from $\mathbf{a}$ by writing

$$a^0_{ij} = a_{ij} \quad (i \neq j), \qquad a^0_{ii} = a_{ii}+\lambda,$$

so that $|\mathbf{a}^0|$, $\|\mathbf{A}^0\|$, and $|\mathbf{b}^0_1|$, which are all elements of $\mathfrak{R}^0$, are in fact monic polynomials in λ with coefficients in $\mathfrak{R}$, and in particular $|\mathbf{b}^0_1|$ is not a zero-divisor in $\mathfrak{R}^0$. Then, since equation (2) is valid over the ring $\mathfrak{R}^0$, we have

$$|\mathbf{b}^0_1|(\|\mathbf{A}^0\| - |\mathbf{a}^0|) = 0,$$

so that $\|\mathbf{A}^0\| = |\mathbf{a}^0|$. Then the natural homomorphism $\mathfrak{R}^0 \to \mathfrak{R}$ yields $\|\mathbf{A}\| = |\mathbf{a}|$ as required.

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ON SOME DUAL INTEGRAL EQUATIONS

By B. NOBLE (*Keele*)

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Summary

A FORMAL solution of the dual integral equations

$$\left.\begin{aligned} \int\limits_0^\infty t^\alpha \phi(t) J_\nu(\rho t)\,dt &= f(\rho) \quad (0 < \rho < 1) \\ \int\limits_0^\infty \phi(t) J_\nu(\rho t)\,dt &= g(\rho) \quad (\rho > 1) \end{aligned}\right\} \tag{1}$$

is obtained when $f(\rho)$, $g(\rho)$ are given, $\phi(t)$ is to be found, and α is a given constant.

1. Introduction

The solution of the dual integral equations (1) when $g(\rho) = 0$ has been given by Titchmarsh [(**2**) 339] for $\alpha > 0$. The solution has been extended to $\alpha > -2$ by Busbridge (**1**). I shall use Busbridge's form of the solution to solve the more general case $g(\rho) \neq 0$ for $-2 < \alpha < 0$. Two methods of extending the solution to a wider range of α will be discussed.

The above problem has been considered by Tranter (**3**) in the particular case $\alpha = \pm 1$. The solutions given by Tranter can be reduced to the simpler solutions given in this paper. References to physical applications of the equations can be found in (**3**).

As in (**3**), the procedure is essentially formal. It will be assumed that the order of integration in repeated integrals, and the order of differentiation and integration, can be reversed as necessary without explicit justification. Solutions can be verified by substitution in the original equations.

2. Solution of (1) for $-2 < \alpha < 0$

Since the equations are linear, the complete solution can be obtained by adding the solutions for $f(\rho) = 0$ and $g(\rho) = 0$. In the following consider the case $f(\rho) = 0$. From the Hankel inversion formula, if we introduce a function $G(t)$ such that

$$\int\limits_0^\infty t\,G(t) J_\nu(\rho t)\,dt = \begin{cases} g(\rho) & (\rho > 1), \\ 0 & (0 < \rho < 1), \end{cases}$$

then
$$G(t) = \int_1^\infty \xi\, g(\xi) J_\nu(\xi t)\, d\xi. \tag{2}$$

Equations (1) with $f(\rho) = 0$ can then be written

$$\left.\begin{aligned} &\int_0^\infty t^\alpha\{\phi(t) - t\,G(t)\}J_\nu(\rho t)\, dt = -\int_0^\infty \lambda^{1+\alpha} G(\lambda) J_\nu(\rho\lambda)\, d\lambda \quad (0 < \rho < 1) \\ &\int_0^\infty \{\phi(t) - t\,G(t)\}J_\nu(\rho t)\, dt = 0 \quad (\rho > 1) \end{aligned}\right\}. \tag{3}$$

But the solution of (1) with $g(\rho) = 0$ is given in (**1**) by

$$\phi(t) = \frac{2^{-\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\left[J_{\nu+\frac{1}{2}\alpha}(t)\int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} f(u)\, du + \right.$$
$$\left. + t\int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha}\, du \int_0^1 f(yu) y^{2+\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(ty)\, dy\right]. \tag{4}$$

Hence the solution of (3) is given by

$$\phi(t) - t\,G(t)$$
$$= -\frac{2^{-\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\left[J_{\nu+\frac{1}{2}\alpha}(t)\int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha}\, du \int_0^\infty \lambda^{1+\alpha} G(\lambda) J_\nu(u\lambda)\, d\lambda + \right.$$
$$\left. + t\int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha}\, du \int_0^1 y^{2+\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(ty)\, dy \int_0^\infty \lambda^{1+\alpha} G(\lambda) J_\nu(yu\lambda)\, d\lambda\right]. \tag{5}$$

Next change the order of integration and evaluate the integral with respect to u in each of the integrals on the right-hand side by means of Sonine's first finite integral [(**4**) 373]

$$\int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} J_\nu(u\lambda)\, du = 2^{\frac{1}{2}\alpha}\Gamma(1+\tfrac{1}{2}\alpha)\lambda^{-1-\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(\lambda).$$

Also substitute the value of $G(\lambda)$ from (2). Then (5) becomes

$$\phi(t) = t\int_1^\infty \xi\, g(\xi) J_\nu(\xi t)\, d\xi - t^{1-\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha}(t)\int_1^\infty \xi\, g(\xi)\, d\xi \int_0^\infty \lambda^{\frac{1}{2}\alpha} J_\nu(\xi\lambda) J_{\nu+1+\frac{1}{2}\alpha}(\lambda)\, d\lambda$$
$$- t^{2-\frac{1}{2}\alpha}\int_0^1 y\, J_{\nu+1+\frac{1}{2}\alpha}(ty)\, dy \int_1^\infty \xi\, g(\xi)\, d\xi \int_0^\infty \lambda^{\frac{1}{2}\alpha} J_\nu(\xi\lambda) J_{\nu+1+\frac{1}{2}\alpha}(\lambda)\, d\lambda. \tag{6}$$

Now $y \leqslant \xi$, and by expressing a hypergeometric function in integral form [(**4**) 401] we find

$$\int\limits_0^\infty \lambda^{\frac{1}{2}\alpha} J_\nu(\xi\lambda) J_{\nu+1+\frac{1}{2}\alpha}(y\lambda)\, d\lambda = \frac{2^{1+\frac{1}{2}\alpha} y^{-\nu-1-\frac{1}{2}\alpha}}{\Gamma(-\frac{1}{2}\alpha)\xi^\nu} \int\limits_0^y z^{2\nu+\alpha+1}(\xi^2-z^2)^{-1-\frac{1}{2}\alpha}\, dz.$$

This is true only for $\alpha < 0$. In this case (6) becomes

$$\phi(t) = t\int\limits_1^\infty \xi\, g(\xi) J_\nu(\xi t)\, d\xi -$$

$$-\frac{2^{1+\frac{1}{2}\alpha} t^{1-\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha}(t)}{\Gamma(-\frac{1}{2}\alpha)} \int\limits_0^\infty \xi^{1-\nu} g(\xi)\, d\xi \int\limits_0^1 z^{2\nu+\alpha+1}(\xi^2-z^2)^{-1-\frac{1}{2}\alpha}\, dz -$$

$$-\frac{2^{1+\frac{1}{2}\alpha} t^{2-\frac{1}{2}\alpha}}{\Gamma(-\frac{1}{2}\alpha)} \int\limits_0^1 y^{-\nu-\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(ty)\, dy \times$$

$$\times \int\limits_1^\infty \xi^{1-\nu} g(\xi)\, d\xi \int\limits_0^y z^{2\nu+\alpha+1}(\xi^2-z^2)^{-1-\frac{1}{2}\alpha}\, dz. \quad (7)$$

In the last integral integrate by parts to find

$$\int\limits_0^1 y^{-\nu-\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(ty)\, dy \int\limits_0^y z^{2\nu+\alpha+1}(\xi^2-z^2)^{-1-\frac{1}{2}\alpha}\, dz$$

$$= -\frac{1}{t} J_{\nu+\frac{1}{2}\alpha}(t) \int\limits_0^1 z^{2\nu+\alpha+1}(\xi^2-z^2)^{-1-\frac{1}{2}\alpha}\, dz +$$

$$+\frac{1}{t} \int\limits_0^1 y^{\nu+\frac{1}{2}\alpha+1} J_{\nu+\frac{1}{2}\alpha}(ty)(\xi^2-y^2)^{-1-\frac{1}{2}\alpha}\, dy.$$

If this is substituted in (7), it will be seen that the second integral cancels with part of the third to give

$$\phi(t) = t\int\limits_1^\infty \xi\, g(\xi) J_\nu(\xi t)\, d\xi -$$

$$-\frac{2^{1+\frac{1}{2}\alpha} t^{1-\frac{1}{2}\alpha}}{\Gamma(-\frac{1}{2}\alpha)} \int\limits_1^\infty \xi^{1-\nu} g(\xi)\, d\xi \int\limits_0^1 y^{\nu+\frac{1}{2}\alpha+1} J_{\nu+\frac{1}{2}\alpha}(ty)(\xi^2-y^2)^{-1-\frac{1}{2}\alpha}\, dy. \quad (8)$$

Next write

$$\int_0^1 y^{\nu+\frac{1}{2}\alpha+1}J_{\nu+\frac{1}{2}\alpha}(ty)(\xi^2-y^2)^{-1-\frac{1}{2}\alpha}\,dy$$
$$= 2^{-1-\frac{1}{2}\alpha}\Gamma(-\tfrac{1}{2}\alpha)\xi^{\nu}t^{\frac{1}{2}\alpha}J_{\nu}(\xi t) - \int_1^{\xi} y^{\nu+\frac{1}{2}\alpha+1}J_{\nu+\frac{1}{2}\alpha}(ty)(\xi^2-y^2)^{-1-\frac{1}{2}\alpha}\,dy, \quad (9)$$

where I have written $\int_0^1 = \int_0^{\xi} - \int_1^{\xi}$ and evaluated the first integral on the right by Sonine's first finite integral. On using this result in (8) we find that the first integral in (8) cancels with part of the second integral and we have finally

$$\phi(t) = \frac{2^{1+\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(-\frac{1}{2}\alpha)} \int_1^{\infty} \xi^{1-\nu}g(\xi)\,d\xi \int_1^{\xi} y^{\nu+\frac{1}{2}\alpha+1}J_{\nu+\frac{1}{2}\alpha}(ty)(\xi^2-y^2)^{-1-\frac{1}{2}\alpha}\,dy. \quad (10)$$

This can be reduced to a form comparable with (4) by changing the order of integration and writing $\xi = yu$. Then

$$\phi(t) = \frac{2^{1+\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(-\frac{1}{2}\alpha)} \int_1^{\infty} u^{1-\nu}(u^2-1)^{-1-\frac{1}{2}\alpha}\,du \int_1^{\infty} y^{1-\frac{1}{2}\alpha}g(yu)J_{\nu+\frac{1}{2}\alpha}(ty)\,dy. \quad (11)$$

This solution holds only for $\alpha < 0$.

As an example suppose that $\nu = 1$, $\alpha = -1$, $g(\rho) = \rho^{-1}$. Then the repeated integral in (11) splits into two separate integrals. The integral in u is elementary. The integral in y reduces to

$$\int_1^{\infty} \sin ty\,dy = \int_0^{\infty} \sin ty\,dy - \int_0^1 \sin ty\,dy = t^{-1}\cos t,$$

where we have taken the infinite integrals to be summable $(C, 1)$ [(**2**) 27]. Finally we find that $\phi(t) = \cos t$, and it is easily verified that this is correct [(**4**) 405].

3. First method of extension of the solution to $\alpha > 0$

Integrate the inner integral in (10) by parts to get

$$\phi(t) = \frac{2^{\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(1-\frac{1}{2}\alpha)}\Bigg[J_{\nu+\frac{1}{2}\alpha}(t)\int_1^{\infty}(\xi^2-1)^{-\frac{1}{2}\alpha}\xi^{1-\nu}g(\xi)\,d\xi +$$
$$+t\int_1^{\infty}\xi^{1-\nu}g(\xi)\,d\xi\int_1^{\xi} y^{\nu+\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha-1}(ty)(\xi^2-y^2)^{-\frac{1}{2}\alpha}\,dy\Bigg]. \quad (12)$$

On changing the order of integration in the second integral and setting $\xi = yu$ we find that

$$\phi(t) = \frac{2^{\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(1-\frac{1}{2}\alpha)}\Big[J_{\nu+\frac{1}{2}\alpha}(t)\int\limits_1^\infty (\xi^2-1)^{-\frac{1}{2}\alpha}\xi^{1-\nu}g(\xi)\,d\xi +$$

$$+t\int\limits_1^\infty y^{2-\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha-1}(ty)\,dy\int\limits_1^\infty u^{1-\nu}g(yu)(u^2-1)^{-\frac{1}{2}\alpha}\,du\Big]. \quad (13)$$

The form of this solution suggests that it should be valid for $\alpha < +2$. It may be noted that, if exactly the same procedure is applied to Titchmarsh's solution [(2) 339] valid for $\alpha > 0$, then Busbridge's solution (4) above, valid for $\alpha > -2$, is obtained.

As an instance, consider the example given by Tranter (3), in which $\nu = 1$, $\alpha = 1$, $g(\rho) = \rho^{-1}$. Then

$$\phi(t) = \frac{2^{\frac{1}{2}}t^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\Big[J_{\frac{3}{2}}(t)\int\limits_1^\infty (\xi^2-1)^{-\frac{1}{2}}\xi^{-1}\,d\xi + t\int\limits_1^\infty y^{\frac{3}{2}}J_{\frac{1}{2}}(ty)\,dy\int\limits_1^\infty (u^2-1)^{-\frac{1}{2}}u^{-1}\,du\Big].$$

All the integrals are elementary and we find that $\phi(t) = t^{-1}\sin t$, which is correct.

4. Second method for extension of the solution to $\alpha > 0$

By using well-known formulae for differentiation and integration of Bessel functions it is possible to change the dual integral equations (1) to an equivalent pair of integral equations with α increased or decreased by 2. Thus multiply both sides of the first equation in (1) by $\rho^{\nu+1}$ and integrate throughout with respect to ρ from 0 to ρ. Multiply both sides of the second equation in (1) by $\rho^{-\nu}$ and differentiate throughout with respect to ρ. This gives

$$\int\limits_0^\infty t^{\alpha-1}\phi(t)J_{\nu+1}(\rho t)\,dt = \rho^{-\nu-1}\int\limits_0^\rho \lambda^{\nu+1}f(\lambda)\,d\lambda \quad (0<\rho<1), \quad (14\,\text{a})$$

$$\int\limits_0^\infty t\,\phi(t)J_{\nu+1}(\rho t)\,dt = -\rho^\nu\frac{d}{d\rho}\{\rho^{-\nu}g(\rho)\} \quad (\rho > 1). \quad (14\,\text{b})$$

For simplicity assume that $f(\rho) = 0$ $(0 < \rho < 1)$. Solve (14) by (10) to find

$$\phi(t) = -\frac{2^{\frac{1}{2}\alpha}t^{1-\frac{1}{2}\alpha}}{\Gamma(1-\frac{1}{2}\alpha)}\int\limits_1^\infty \frac{d}{d\xi}\{\xi^{-\nu}g(\xi)\}\,d\xi\int\limits_1^\xi y^{\nu+\frac{1}{2}\alpha+1}J_{\nu+\frac{1}{2}\alpha}(ty)(\xi^2-y^2)^{-\frac{1}{2}\alpha}\,dy.$$

This solution can be reduced to the form (13) by writing the integrand of the inner integral as

$$-\tfrac{1}{2}(1-\tfrac{1}{2}\alpha)^{-1}y^{\nu+\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha}(ty)\frac{d}{dy}(\xi^2-y^2)^{1-\frac{1}{2}\alpha}$$

and integrating by parts: then change the order of integration and integrate the new inner integral by parts.

Another type of solution can be obtained by multiplying the first equation in (1) by $\rho^{-\nu+1}$ and integrating, and multiplying the second equation by ρ^{ν} and differentiating. We find

$$\int\limits_0^\infty t^{\alpha-1}\phi(t)J_{\nu-1}(\rho t)\,dt = -\rho^{\nu-1}\int\limits_0^\rho \lambda^{-\nu+1}f(\lambda)\,d\lambda+\rho^{\nu-1}C \quad (0<\rho<1), \tag{15 a}$$

$$\int\limits_0^\infty t\,\phi(t)J_{\nu-1}(\rho t)\,dt = \rho^{-\nu}\frac{d}{d\rho}\{\rho^{\nu}g(\rho)\} \quad (\rho>1), \tag{15 b}$$

where C is a constant which can be expressed as an integral over $\phi(t)$, but this does not help since $\phi(t)$ is unknown. In order to find an explicit expression for C, solve

$$\int\limits_0^\infty t^{\alpha-1}\phi(t)J_{\nu-1}(\rho t)\,dt = \rho^{\nu-1}C \quad (0<\rho<1),$$

$$\int\limits_0^\infty t\,\phi(t)J_{\nu-1}(\rho t)\,dt = 0 \qquad (\rho>1).$$

On using (4) the solution is found to be

$$\phi(t) = C\frac{2^{1-\frac{1}{2}\alpha}\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2}\alpha-1)}t^{-\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha-1}(t). \tag{16}$$

Substitute in the second equation in (1) to get

$$\int\limits_0^\infty \phi(t)J_{\nu}(\rho t)\,dt = C\frac{2^{1-\alpha}\{\Gamma(\nu)\}^2}{\Gamma(\nu+\frac{1}{2}\alpha)\Gamma(\nu+\frac{1}{2}\alpha-1)}\rho^{-\nu}.$$

Hence, if the coefficient of $\rho^{-\nu}$ in $g(\rho)$ is k, then

$$C = k\frac{\Gamma(\nu+\frac{1}{2}\alpha)\Gamma(\nu+\frac{1}{2}\alpha-1)}{2^{1-\frac{1}{2}\alpha}\{\Gamma(\nu)\}^2},$$

and the contribution of C to the solution is

$$\phi(t) = k\frac{2^{\frac{1}{2}\alpha}\Gamma(\nu+\frac{1}{2}\alpha)}{\Gamma(\nu)}t^{-\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha-1}(t). \tag{17}$$

We again illustrate with Tranter's example, $\nu = 1$, $\alpha = 1$, $g(\rho) = \rho^{-1}$. Then $k = 1$, and (17) gives directly $\phi(t) = t^{-1}\sin t$, which is correct.

In this section I have changed the dual integral equations into an equivalent pair by integrating the first equation and differentiating the second. In a similar way by differentiating the first and integrating the second it is possible to increase the value of α by 2. I shall not pursue this matter further here.

5. Equivalence of the above solutions and Tranter's solutions

Inspection of Tranter's paper (**3**) will show that his solutions are fourfold integrals, whereas the solutions given here [e.g. (11) above] contain only two integrations. It is possible to reduce Tranter's fourfold integrals by performing two of the integrations explicitly. This can be done by judicious use of integration by parts to allow interchange of orders of integration, and use of Sonine's integral [(**4**) 373] and the Weber–Schafheitlin integral [(**4**) 401]. The manipulations are tricky, but present few points of mathematical interest. Details are therefore omitted.

6. Acknowledgement

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NOTE ON INTEGRAL FUNCTIONS OF INFINITE ORDER

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1. SHAH (**2**) has shown that, if $f(z)$ is an integral function of infinite order, then
$$\liminf_{r\to\infty}\{\log M(r,f)\}/N(r,f) = 0,$$
where $M(r,f) = \sup_{|z|=r}|f(z)|$ and $N(r,f)$ is the central index of $f(z)$ for $|z| = r$. Later he and Kharma (**3**) showed that the same is true if $M(r,f)$ is replaced by $rM(r,f')$. My purpose in this brief note is to prove that
$$\liminf_{r\to\infty}\{\log r^{p}M(r,f^{(p)})\}/N(r,f) = 0,$$
where p is any function of N such that $p(N) = o(N/\log N)$.

2. The preliminary results are given in a number of lemmas.

LEMMA 1. (**1**) *If $\mu(r,f)$ is the maximum term of $f(z)$ for $|z| = r$, then*
$$\liminf_{r\to\infty}\{\log \mu(r,f)\}/N(r,f) = 0.$$

Let $\sigma(x) = \sup_{1\leqslant r\leqslant x}\{\log N(r,f)\}/\log r$. It is known [(**4**) 33] that $\sigma(x)\to\infty$ as $x\to\infty$, and consequently, for an infinite sequence of x,
$$\sigma(x) = \{\log N(x,f)\}/\log x.$$
For an x of this sequence we have [(**4**) 30]
$$\begin{aligned}\log \mu(x,f) &= K + \int_1^x N(r,f)/r\,dr\\ &\leqslant K + \int_1^x r^{\sigma(x)-1}\,dr\\ &\leqslant K + x^{\sigma(x)}/\sigma(x)\\ &= K + N(x,f)/\sigma(x),\end{aligned}$$
and so the lemma follows.

Quart. J. Math. Oxford (2), 6 (1955), 88–90.

LEMMA 2. *If* $f(z) = \sum_0^\infty a_n z^n$, *then* $\phi(z) = \sum_1^\infty (a_n/n)z^n$ *is an integral function whose central indices are a sub-sequence of those of* $f(z)$. *If* $p(N) = o(N)$, *then, for some arbitrarily large* ρ,

$$\rho^p M(\rho, f^{(p)}) = O\{N^{2p}\mu(\rho, f)\},$$

where $N = N(\rho, f)$ *is also a central index of* $\phi(z)$.

It is obvious that $\phi(z)$ is an integral function, and so given r we have, with $N = N(r, \phi)$,

$$(|a_n|/n)r^n \leqslant (|a_N|/N)r^N,$$

i.e.

$$(|a_n|/r^n)/(|a_N|/r^N) \leqslant n/N.$$

Choose R so that $nR^n \leqslant NR^N$, which is possible for all N, and we get

$$\{|a_n|/(rR)^n\}/\{|a_N|/(rR)^N\} \leqslant (nR^n)/(NR^N) \leqslant 1, \tag{1}$$

and so, if $F(z) = \sum_1^\infty nz^n$,

$$\mu(rR, f) = |a_N|(rR)^N = \mu(r, \phi)\mu(R, F), \tag{2}$$

which shows that the central indices of $\phi(z)$ form a sub-sequence of those of $f(z)$. From (1) it follows that

$$\{n(n-1)...(n-p+1)|a_n|(rR)^n\}/\{|a_N|(rR)^N\}$$
$$\leqslant \{n(n-1)...(n-p+1)nR^n\}/\{NR^N\}$$

and thus the final part of the lemma will follow if

$$\sum_1^\infty n(n-1)...(n-p+1)nR^n = O\{N^{2p}\mu(R, F)\}.$$

It is not difficult to show that

$$\sum_1^\infty n(n-1)...(n-p+1)nR^n = (p!\,R^p)/(1-R)^{p+1}.$$

Now, if ξ is the value of x for which $\log x - x\log(1/R)$ is a maximum, i.e. $1/\xi = \log 1/R$, so that $|\xi - N| < 1$, then, with $p(N) = o(N)$,

$$(p!\,R^p)/(1-R)^{p+1} = O(p^p\xi^{p+1}) = O\{N^{2p}\mu(R, F)\},$$

and so the lemma follows.

LEMMA 3. *If* $N(r, \phi) = N(R, F) = N(rR, f)$, *then*

$$\liminf_{r\to\infty}\{\log \mu(rR, f)\}/N(rR, f) = 0.$$

This is a direct consequence of (2) since, by Lemma 1,

$$\liminf_{r\to\infty}\{\log \mu(r, \phi)\}/N(r, \phi) = 0.$$

The result now follows immediately, for, choosing r to be a value of $|z|$ to which Lemma 3 applies, we obtain

$$\liminf_{r\to\infty}\{\log r^p M(r,f^{(p)})\}/N(r,f)$$
$$\leqslant \lim_{N\to\infty}(2p\log N)/N+\liminf_{r\to\infty}\{\log\mu(r,f)\}/N(r,f),$$

and this is zero by the choice of p and Lemma 3.

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OBSTRUCTIONS TO COMPRESSION

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1. Introduction

THE theory of obstructions to extensions of mappings (**3**) has simplified and unified many results in homotopy theory. It has proved to be a natural tool in such questions (**5**), (**9**), (**11**) and leads to algebraic invariants of maps, these invariants being described in terms of cohomology classes of the antecedent space with coefficients in the homotopy groups of the image space.

In this note we consider a 'dual' theory in which the problem is to deform a map $f\colon (X, A) \to (Y, B)$ into one with values in a subspace Y' $(B \subset Y')$ by a homotopy of the form† $(X, A) \to (Y, B)$. We shall refer to this as the problem of *compressing f into Y'*. When the spaces considered are suitably restricted, this leads to obstructions which are cycles or homology classes in (Y, B) with cohomotopy groups of (X, A) as coefficients. Such obstructions are considered in (**8**). Here we use different methods and obtain results which are, in some ways, sharper than any of those in (**8**).

Let X be a CW-complex (**10**) and let A be a sub-complex of X. Then $X \times 1 \cup A \times I$ is a (strong) deformation retract of $X \times I$. Therefore we have [cf. (1.4) in (**8**)]:

LEMMA 1.1. *If $f \simeq g\colon (X, A) \to (Y, B)$, where $gX \subset Y'$ $(B \subset Y' \subset Y)$, then $f \simeq h$, rel A, where $hX \subset Y'$.*

We use S^n to stand for the n-sphere in Hilbert space consisting of all points $(t_0, t_1,...)$ such that $t_0^2+t_1^2+... = 1$ and $t_i = 0$ if $i > n$. We write $(1, 0, 0,...) = z_0$ and E^n will denote the hemisphere of S^n in which $t_n \geqslant 0$. We orient‡ S^n, E^n inductively in such a way that the orientation of E^n is coherent with that of S^{n-1} and determines the orientation of S^n. We take S^n, with base-point z_0, to be the image space in the definition of cohomotopy groups (**7**).

† Except when the contrary is stated, a homotopy of a map $(X, A) \to (Y, B)$ will always mean one of this form.

‡ i.e. we choose generators of $H_n(S^n)$, $H_n(E^n, \dot{E}^n)$.

2. Statement of results

Let X, Y be CW-complexes and A, B (possibly empty) sub-complexes of X, Y. We write

$$X^n \cup A = \bar{X}^n, \qquad Y^n \cup B = \bar{Y}^n$$

and we assume that, for each $n \geqslant 0$, there are but a finite number of (open) n-cells in $Y-B$. If $\dim(X-A) \leqslant 2n-2$ (i.e. if $X = \bar{X}^{2n-2}$), then the cohomotopy group $\pi^n(X, A)$ is defined by the methods used in (**7**). A similar remark applies to $\pi^n(Y, B)$ and we identify $\pi^n(\bar{Y}^n, \bar{Y}^{n-1})$ $(n \geqslant 1)$ with the group $C^n(Y, B)$ of integral n-cochains in (Y, B) [(**7**), § 11], thus defining a group structure in the set $\pi^n(\bar{Y}^n, \bar{Y}^{n-1})$ if $n = 1$. We define $\delta\colon C^n(Y, B) \to C^{n+1}(Y, B)$ as the coboundary homomorphism in the cohomotopy sequence of the triple $(\bar{Y}^{n+1}, \bar{Y}^n, \bar{Y}^{n-1})$. We define the group of n-chains, with coefficients in G, to be

$$C_n(Y, B; G) = \operatorname{Hom}(C^n(Y, B), G)$$

and $\partial\colon C_{n+1}(Y, B; G) \to C_n(Y, B; G)$ by $\partial c = c\delta$.

Let $k \geqslant 2$, $\dim(X-A) \leqslant 2k-2$, and let $f\colon (X, A) \to (\bar{Y}^k, B)$ be a given map. Let†

$$z_k(f) = f^{\#} j^{\#} \in C_k(Y, B; \pi^k(X, A)),$$

where $j\colon (\bar{Y}^k, B) \subset (\bar{Y}^k, \bar{Y}^{k-1})$. We have $j = j_2 j_1$, where j_1, j_2 are the inclusion maps

$$(\bar{Y}^k, B) \xrightarrow{j_1} (\bar{Y}^k, \bar{Y}^{k-2}) \xrightarrow{j_2} (\bar{Y}^k, \bar{Y}^{k-1}),$$

and $j_2^{\#} \delta \pi^{k-1}(\bar{Y}^{k-1}, \bar{Y}^{k-2}) = 0$. Therefore $j^{\#}\delta = 0$, and it follows that $\partial z_k(f) = 0$. Obviously $z_k(f) = z_k(g)$ if $f \simeq g$; also $z_k(g) = 0$ if $gX \subset \bar{Y}^{k-1}$. Therefore‡ $z_k(f) = 0$ if f is compressible into $\bar{Y}^{k-1}$. We describe $z_k(f)$ as the *first obstruction to compressing f into $\bar{Y}^{k-1}$*. In § 4 below we prove:

THEOREM 2.1. *Let $\bar{Y}^{k-1}$ be m-connected, where $m \leqslant k-1$, and let $\dim(X-A) \leqslant k+m-1$. Then f is compressible into $\bar{Y}^{k-1}$ if $z_k(f) = 0$.*

Let $T = X \times 0 \cup A \times I \cup X \times 1 \subset X \times I$ and let

$$F\colon (X \times I, T, A \times I) \to (\bar{Y}^{k+1}, \bar{Y}^k, B) \tag{2.2}$$

be a homotopy in $\bar{Y}^{k+1}$ between two maps

$$f_i\colon (X, A) \to (\bar{Y}^k, B) \quad (f_i x = F(x, i);\ i = 0, 1).$$

Then F induces a homomorphism

$$F^{\#} \in C_{k+1}(Y, B; \pi^{k+1}(X \times I, T)).$$

† $g^{\#}$ will always denote the homomorphism of the appropriate cohomotopy group which is induced by a given map g.

‡ In general the converse is not true (e.g. consider an essential map $S^k \to P^k$, where k is even and P^k is real projective k-space).

Let $X_I = X \times I$, $A_I = A \times I$ and consider the stretch

$$\pi^k(X_I, A_I) \xrightarrow{i^\#} \pi^k(T, A_I) \xrightarrow{\Delta} \pi^{k+1}(X_I, T) \xrightarrow{j^\#} \pi^{k+1}(X_I, A_I)$$

in the cohomotopy sequence of the triple (X_I, T, A_I). Let

$$\omega_i\colon (X, A) \to (T, A_I)$$

be defined by $\omega_i x = (x, i)$ and let

$$\pi^k(T, A_I) \xrightarrow{\theta} \pi^k(X, A) + \pi^k(X, A) \xrightarrow{\phi} \pi^k(X, A)$$

be defined by $\theta\alpha = (\omega_0^\# \alpha, \omega_1^\# \alpha)$, $\phi(\beta_0, \beta_1) = \beta_1 - \beta_0$, where $+$ indicates direct summation. Obviously θ is an isomorphism onto and $\pi^k(X_I, A_I)$ may be identified with $\pi^k(X, A)$ in such a way that $\theta i^\# \beta = (\beta, \beta)$. Thus kernel $i^\# =$ image $j^\# = 0$, whence Δ is onto. Also ϕ is onto and

$$\theta\Delta^{-1}(0) = \phi^{-1}(0).$$

Hence it follows that

$$\phi\theta\Delta^{-1}\colon \pi^{k+1}(X_I, T) \approx \pi^k(X, A).$$

Let $\rho = \phi\theta\Delta^{-1}$ and let

$$d_{k+1}(F) = \rho F^\# \in C_{k+1}(Y, B; \pi^k(X, A)). \tag{2.3}$$

Evidently $C_{k+1}(Y, B; G) = C_{k+1}(Y, \overline{Y}^k; G)$ and $F^\# = z_{k+1}(F')$, where

$$F'\colon (X_I, T) \to (\overline{Y}^{k+1}, \overline{Y}^k)$$

is the map determined by F. Therefore $d_{k+1}(F) = 0$ if F' is compressible into $\overline{Y}^k$. Obviously

$$\dim(X_I - T) \leqslant \dim(X - A) + 1$$

and, since ρ is an isomorphism, it follows from Lemma 1.1 and Theorem 2.1, with k replaced by $k+1$, that, if $\overline{Y}^k$ is m-connected ($m \leqslant k$) and $\dim(X - A) \leqslant k + m - 1$, then $d_{k+1}(F) = 0$ implies that F is homotopic, rel T, to a map in $\overline{Y}^k$. Notice that, if $m \leqslant k-1$ and $\overline{Y}^{k-1}$ is m-connected, so is $\overline{Y}^k$.

Let F_1 be a homotopy of the form (2.2) such that $F_1(x, 0) = F(x, 1)$. Then a homotopy $F + F_1$, of the form (2.2), is defined by

$$(F + F_1)(x, t) = F(x, 2t) \quad \text{or} \quad F_1(x, 2t-1)$$

according as $t \leqslant \frac{1}{2}$ or $t \geqslant \frac{1}{2}$. We shall prove

$$\partial d_{k+1}(F) = z_k(f_1) - z_k(f_0), \tag{2.4}$$

$$d_{k+1}(F + F_1) = d_{k+1}(F) + d_{k+1}(F_1). \tag{2.5}$$

Let $0 \leqslant m \leqslant k$ and assume that

[2.6] *every map* $(X, A) \to (\overline{Y}^k, B)$ *is homotopic, in* $(\overline{Y}^k, B)$, *to one which is constant on* X^m.

Clearly (2.6) is true if $\overline{Y}^k$ is m-connected and either A or B is m-connected (or empty). Subject to (2.6) we shall prove

[2.7] *if* $\dim(X-A) \leqslant k+m-1$ *and if* $f\colon (X, A) \to (\overline{Y}^k, B)$ *and*
$$c \in C_{k+1}(Y, B; \pi^k(X, A))$$
are given, then there is a homotopy F, *of the form* (2.2), *such that* $F(x, 0) = fx$ *and* $d_{k+1}(F) = c$.

Let $\overline{Y}^{k-1}$ be m-connected ($m \leqslant k-1$), let (2.6) be true, let
$$\dim(X-A) \leqslant k+m-1$$
and let $f\colon (X, A) \to (\overline{Y}^k, B)$ be given. Let $\{z_k(f)\} \in H_k(Y, B; \pi^k(X, A))$ be the homology class of $z_k(f)$ and let f be related by a homotopy F, of the form (2.2), to a map g in $\overline{Y}^{k-1}$. Then $z_k(g) = 0$, and it follows from (2.4) that $\{z_k(f)\} = 0$. Conversely, let $\{z_k(f)\} = 0$. Then $z_k(f) = -\partial c$, for some $c \in C_{k+1}(Y, B; \pi^k(X, A))$. Let F be as in (2.7) and let $f \simeq g$ by the homotopy F. Then it follows from (2.4) that $z_k(g) = \partial c + z_k(f) = 0$. Therefore it follows from (2.1) that g is compressible into $\overline{Y}^{k-1}$. Thus there is a homotopy $g_t\colon (X, A) \to (\overline{Y}^{k+1}, B)$ such that $g_0 x = fx, g_1 X \subset \overline{Y}^{k-1}$. Hence, and from (1.1), we have

THEOREM 2.8. *A map* $f\colon (X, A) \to (\overline{Y}^k, B)$ *is homotopic in* $\overline{Y}^{k+1}$, rel A, *to a map in* $\overline{Y}^{k-1}$ *if and only if* $\{z_k(f)\} = 0$.

Let A, B be empty and X finite. Let $\dim X \leqslant k$ and let Y be 1-connected. Since $\dim X \leqslant k$, we may identify $H_k(X; G)$ with the group of (k, G)-cycles in X, for any coefficient group G. Therefore we have
$$z_k(i) = j^\# \in H_k(X; \pi^k(X)),$$
where $i\colon X \subset X$, $j\colon X \subset (X, X^{k-1})$. Let $f\colon X \to Y$ be a given map, which we may assume to be cellular. Then f determines a map
$$g\colon (X, X^{k-1}) \to (Y^k, Y^{k-1}).$$
The diagram
$$\begin{tikzcd} \pi^k(Y^k, Y^{k-1}) \arrow[r, "j^\#"] \arrow[d, "g^\#"] & \pi^k(Y^k) \arrow[d, "g^\#"] \\ \pi^k(X, X^{k-1}) \arrow[r, "j^\#"] & \pi^k(X) \end{tikzcd}$$
is commutative and $z_k(f) = g^\# j^\# = j^\# g^\# = g_\# z_k(i)$, where
$$g_\#\colon C_k(X; \pi^k(X)) \to C_k(Y; \pi^k(X))$$
is the homomorphism induced by g. Hence it follows that
$$f_* z_k(i) = \{z_k(f)\},$$
where
$$f_*\colon H_k(X; \pi^k(X)) \to H_k(Y; \pi^k(X))$$

is the homomorphism induced by f. Since Y is 1-connected, it is of the same homotopy type as a cell-complex Y_2 which has no 1-cells. Let $u: Y \to Y_2$, $v: Y_2 \to Y$ be cellular homotopy equivalences such that $vu \simeq 1$. If $uf \simeq h$, where $hX \subset Y_2^{k-1}$, then $f \simeq vh$ and $vhX \subset Y^{k-1}$. Therefore we may assume that Y has no 1-cells, in which case Y^{k-1} is 1-connected, even if $k = 2$. Moreover, if $f \simeq f'$ in Y, where $f'X \subset Y^{k-1}$, then $f \simeq f'$ in Y^{k+1} since $\dim X \leqslant k$. Therefore, as a corollary of Theorem 2.8, we have a theorem due to Pontryagin (**6**):

THEOREM 2.9. *The map $f: X \to Y$ is compressible into Y^{k-1} if and only if $f_* z_k(i) = 0$.*

Notice that Theorem 2.9 is trivially true if $k = 1$ ($\pi^1(X) = H^1(X)$).

The groups $C_n(Y, B; G)$, the boundary operator ∂, and the corresponding 'cocohomology' groups $H_n(Y, B; G)$ may be defined as above even if $Y - B$ contains infinitely many q-cells, for some or all values of q. If $H_k(X; G)$ is defined in this way, the restriction that X be finite is evidently unnecessary in Theorem 2.9. However, the corresponding restriction on $Y - B$ seems to be essential for our methods.

The limiting technique described in (**7**) can be used to prove similar results when X is any compact space [see also (**1**) for the case of a paracompact X].

Another extension of the theory is to consider the general type of relativization described in (**8**). The theorems proved here may be extended in this direction under a separate assumption of connectedness for each sub-set of Y which occurs in the problem [cf. §§ 6, 7 of (**8**)]. The resulting obstructions will be expressed in terms of homology theory defined in terms of 'stacks' [*faisceaux*; cf. § 13 in (**8**)].

3. The addition of maps

Let X, X' be CW-complexes and $f: X \to X'$ a given map. By a *short homotopy*, $f_t: X \to X'$, we mean one such that, for every sub-complex $X_0 \subset X$ and every $t \in I$, $f_t X_0$ is contained in the smallest sub-complex of X' which contains $f_0 X_0$. We describe a map $g: X \to X'$ as a *cellular approximation* to f if and only if it is cellular and there is a short homotopy f_t with $f_0 = f$, $f_1 = g$. It follows from the (inductive) proof of (L) [(**10**) 229] that every map $f: X \to X'$ has a cellular approximation. Similarly we define a cellular approximation, $g: (X, A) \to (X', A')$, to a map $f: (X, A) \to (X', A')$ if A, A' are sub-complexes of X, X'.

Now let (X, A), (Y, B) be as in § 2, let y_0 be a 0-cell† of B and let X_0

† If the given B is empty, we may obviously replace it by a 0-cell of Y.

be any sub-complex of X. Let $f, g\colon (X, X_0) \to (Y, B)$ be maps such that $f\bar{X}^p = gX^q = y_0$ ($\bar{X}^p = X^p \cup A$; $p, q \geqslant -1$) and let

$$\dim(X-A) \leqslant p+q+1.$$

Define $$f\times g\colon (X\times X, X_0\times X_0) \to (Y\times Y, B\times B)$$

by $(f\times g)(x,x') = (fx, gx')$. If $Y' \subset Y$, let

$$Y'\vee Y' = Y'\times y_0 \cup y_0 \times Y' \subset Y\times Y$$

and define $\Omega\colon (Y\vee Y, B\vee B) \to (Y, B)$ by $\Omega(y, y_0) = \Omega(y_0, y) = y$. Let

$$D\colon (X, X_0) \to (X\times X, X_0\times X_0)$$

be a cellular approximation to the diagonal map $x \to (x, x)$. Then $DA \subset A\times A$ and, since $\dim(X-A) \leqslant p+q+1$,

$$D(X-A) \subset X\times X^q \cup X^p \times X.$$

Therefore $DX \subset X\times X^q \cup \bar{X}^p \times X$ and

$$(f\times g)DX \subset Y\vee Y, \qquad (f\times g)DX_0 \subset (Y\vee Y) \cap (B\times B) = B\vee B.$$

Let $\phi\colon (X, X_0) \to (Y\vee Y, B\vee B)$ be the map determined by $(f\times g)D$ and let

$$f+_D g = \Omega\phi\colon (X, X_0) \to (Y, B). \tag{3.1}$$

We describe $f+_D g$ as a† **D**-*sum* of f, g and assert that it has the properties:

[3.2] *if* $f_t, g_t\colon (X, X_0) \to (Y, B)$ *are homotopies such that* $f_t\bar{X}^p = g_t X^q = y_0$, *then* $f_0+_D g_0 \simeq f_1+_D g_1$ *by the homotopy* $f_t+_D g_t$,

[3.3] *if* $fX = y_0$, *then* $f+_D g \simeq g$,

[3.4] $(f+_D g)^\# = f^\# + g^\#\colon \pi^n(Y, B) \to \pi^n(X, X_0)$ *if*

$$\dim(X-X_0),\ \dim(Y-B) \leqslant 2n-2.$$

Proofs. Here (3.2) is obvious. To prove (3.3) let

$$D_t\colon (X, X_0) \to (X\times X, X_0\times X_0)$$

be a short homotopy between the diagonal map D_0 and $D = D_1$. Let $fX = y_0$. Then $\Omega(f\times g)D_0 x = gx$, for every $x \in X$, and

$$(f\times g)D_t X \subset Y\vee Y, \qquad (f\times g)D_t X_0 \subset B\vee B.$$

Let $\phi_t\colon (X, X_0) \to (Y\vee Y, B\vee B)$ be the homotopy determined by $(f\times g)D_t$. Then $f+_D g \simeq g$ by the homotopy $\Omega\phi_{1-t}$.

In order to prove (3.4) let $u\colon (Y, B) \to (S^n, z_0)$ be given and let $\{v\} \in \pi^n(P, Q)$ denote the homotopy class of a given map

$$v\colon (P, Q) \to (S^n, z_0) \quad (Q \subset P).$$

† Let D' be any other cellular approximation to the diagonal map. Then $f+_D g \simeq f+_{D'} g$ if $\dim(X-A) \leqslant p+q$.

Then $h^{\#}\{u\} = \{uh\}$ $(h = f, g)$ and from (**7**)

$$\{uf +_D ug\} = \{uf\} + \{ug\}. \tag{3.5}$$

Let $\Omega' \colon (S^n \vee S^n, z_0 \vee z_0) \to (S^n, z_0)$ be defined in the same way as Ω and let

$$u \vee u \colon (Y \vee Y, B \vee B) \to (S^n \vee S^n, z_0 \vee z_0)$$

be the map determined by $u \times u$. Then $u\Omega = \Omega'(u \vee u)$. Therefore $u(f +_D g) = uf +_D ug$ and (3.4) follows from (3.5).

Let X_i, Y_i $(i = 1, 2)$ be sub-complexes of X, Y such that $X_1 \subset X_2$, $y_0 \in Y_1 \subset Y_2$. Let $h' \colon (X_2, X_1) \to (Y_2, Y_1)$ denote the map determined by a given map $h \colon (X, X_0) \to (Y, B)$ such that $hX_i \subset Y_i$. Then the following is a straightforward consequence of (3.1):

[3.6] *If fX_i, $gX_i \subset Y_i$, then $(f +_D g)X_i \subset Y_i$ and $(f +_D g)' = f' +_{D'} g'$, where*

$$D' \colon (X_2, X_1) \to (X_2 \times X_2, X_1 \times X_1)$$

is the map determined by D.

4. Proof of Theorem 2.1

We assume, as we obviously may, that $Y = \overline{Y}^k$; also that $Y \neq \overline{Y}^{k-1}$ and $m > 0$ since (2.1) is trivial if $Y = \overline{Y}^{k-1}$ or $m = 0$ (if $m = 0$, then $\dim(X - A) \leqslant k - 1$). Assume the theorem in case $Y - B$ consists of a single k-cell and let $Y_0 = Y - e^k$, where e^k is a k-cell in $Y - B$. Since $m \leqslant k - 1$ and $\overline{Y}^{k-1}$ is m-connected, so is Y_0. Therefore it follows from the special case, with B replaced by Y_0, and from (1.1), that f is compressible into Y_0 by a homotopy rel A. Therefore Theorem 2.1 follows by induction on the number of k-cells in $Y - B$. So we assume that $Y - B$ is a single k-cell, to which we assign an orientation. Then $B = \overline{Y}^{k-1}$ and is m-connected $(m > 0)$. We also assume, as we obviously may, that $f\overline{X}^{k-1} \subset B$.

Let $k - 2 \leqslant p < \dim(X - A)$ and assume that $f \simeq g'$, where $g'\overline{X}^p \subset B$. Let $g \colon (X, \overline{X}^p) \to (Y, B)$ be the map determined by g' and assume that

$$g^{\#}\pi^k(Y, B) = 0 \in \pi^k(X, \overline{X}^p). \tag{4.1}$$

Let $u \colon (Y, B) \to (S^k, z_0)$ be such that $u \mid Y - B$ is of degree 1. Then it follows from (4.1) that $ug \simeq z_0^*$ where $z_0^* X = z_0$. Since (Y, B), B are $(k-1)$-connected, m-connected, and $p + 1 \leqslant \dim(X - A) \leqslant k + m - 1$, it follows from Theorem II in (**2**) that u induces an isomorphism

$$\pi_{p+1}(Y, B) \approx \pi_{p+1}(S^k)$$

and hence an isomorphism

$$u_2 \colon C^{p+1}(X, A; \pi_{p+1}(Y, B)) \approx C^{p+1}(X, A; \pi_{p+1}(S^k)).$$

Let $c \in C^{p+1}(X, A; \pi_{p+1}(Y, B))$ be the obstruction to a homotopy $g \simeq g_1$ such that $g_1 \bar{X}^{p+1} \subset B$. Then $u_* c = 0$ since $ug \simeq z_0^*$. Therefore $c = 0$, and it follows that $g \simeq g_1$, where $g_1 \bar{X}^{p+1} \subset B$.

We proceed to show that $g_1 \simeq g_2$, where $g_2 \bar{X}^{p+1} \subset B$ and the map $(X, \bar{X}^{p+1}) \to (Y, B)$ determined by g_2 satisfies (4.1) with p replaced by $p+1$. Let $h\colon (X, \bar{X}^{p+1}) \to (Y, B)$ be the map determined by g_1: that is to say, $hi = g_1$, where $i\colon (X, \bar{X}^p) \subset (X, \bar{X}^{p+1})$. The element $\{u\}$ generates the (cyclic infinite) group $\pi^k(Y, B)$. Since $g_1 \simeq g$, it follows from (4.1) that

$$i^{\#} h^{\#}\{u\} = g_1^{\#}\{u\} = g^{\#}\{u\} = 0.$$

Therefore $h^{\#}\{u\} = -\Delta\{v\}$, where $v\colon (\bar{X}^{p+1}, \bar{X}^p) \to (S^{k-1}, z_0)$ and

$$\Delta\colon \pi^{k-1}(\bar{X}^{p+1}, \bar{X}^p) \to \pi^k(X, \bar{X}^{p+1})$$

is the coboundary homomorphism in the cohomotopy sequence of the triple $(X, \bar{X}^{p+1}, \bar{X}^p)$. Let

$$(X, \bar{X}^{p+1}) \xrightarrow{w} (E^k, S^{k-1}) \xrightarrow{\psi} (Y, B)$$

be, respectively, an extension of v and a map such that $\psi z_0 = y_0$, which covers $Y - B$ with degree 1. Then $u\psi\colon (E^k, S^{k-1}) \to (S^k, z_0)$ is of degree 1 and

$$\Delta\{v\} = \{u\psi w\} = (\psi w)^{\#}\{u\}. \tag{4.2}$$

Since B is m-connected, we may assume that $hX^m = y_0$. We also have $\psi w \bar{X}^p = y_0$ and $\dim(X - A) \leqslant k + m - 1 \leqslant p + m + 1$. Therefore $\psi w +_D h$ may be defined as in § 3 and

$$(\psi w +_D h)^{\#}\{u\} = \Delta\{v\} + h^{\#}\{u\} = 0$$

in consequence of (3.4) and (4.2). Therefore $\psi w +_D h$ satisfies (4.1) with p replaced by $p+1$. Let $g''\colon (X, \bar{X}^p) \to (Y, B)$ be the map determined by ψw. Then it follows from (3.6) that $\psi w +_D h$ determines a **D**-sum, $g_2\colon (X, \bar{X}^p) \to (Y, B)$, of g'' and g_1 ($g_1 x = hx$). Obviously $g'' \simeq y_0^*$, where $y_0^* X = y_0$, and it follows from (3.2), (3.3) that $g_1 \simeq g_2$.

Since $\dim(Y - B) = k$, the condition (4.1) is satisfied when $p = k-2$, $g' = f$, $z_k(f) = 0$. Therefore Theorem 2.1 follows by induction on p.

5. Proof of (2.4), (2.5), (2.7)

Consider the (commutative) diagram

$$\begin{array}{ccc}
\pi^{k+1}(\bar{Y}^{k+1}, \bar{Y}^k) & \xrightarrow{\rho F^{\#}} & \pi^k(X, A) \\
\uparrow{\scriptstyle \Delta} & & \uparrow{\scriptstyle \rho\Delta = \phi\theta} \\
\pi^k(\bar{Y}^k, \bar{Y}^{k-1}) & \xrightarrow{F^{\#}} & \pi^k(T, A_I),
\end{array}$$

where ρ means the same as in (2.3) and $\bar{F}: (T, A_I) \to (\bar{Y}^k, \bar{Y}^{k-1})$ is the map determined by F. Let $\bar{f}_i: (X, A) \to (\bar{Y}^k, \bar{Y}^{k-1})$ be the map determined by $f_i: (X, A) \to (Y, B)$ $(i = 0, 1)$. Then $\bar{f}_i^\# = z_k(f_i)$ and, obviously, $\omega_i^\# \bar{F}^\# = \bar{f}_i^\#$. Therefore $\phi\theta\bar{F}^\# = z_k(f_1) - z_k(f_0)$ and

$$\partial d_{k+1}(F) = \rho F^\# \Delta = \phi\theta\bar{F}^\# = z_k(f_1) - z_k(f_0).$$

This proves (2.4).

In order to prove (2.5) it is enough to prove that $(F+F_1)^\# = F^\# + F_1^\#$. Let

$$G,\ G_1: (X_I, T, A_I) \to (\bar{Y}^{k+1}, \bar{Y}^k, B)$$

be defined by

$$G(x,t) = F(x, 2t), \qquad G_1(x,t) = F(x,1) \qquad \text{if } t \leqslant \tfrac{1}{2}$$

$$G(x,t) = F(x,1), \qquad G_1(x,t) = F_1(x, 2t-1) \qquad \text{if } t \geqslant \tfrac{1}{2}.$$

Obviously $G \simeq F$, $G_1 \simeq F_1$ and $(F+F_1)(x,t) = G(x,t)$ or $G_1(x,t)$ according as $t \leqslant \frac{1}{2}$ or $t \geqslant \frac{1}{2}$. Let $w: (\bar{Y}^{k+1}, \bar{Y}^k) \to (S^{k+1}, z_0)$ be given. Then $wG(x,t) = z_0$ if $t \geqslant \frac{1}{2}$, $wG_1(x,t) = z_0$ if $t \leqslant \frac{1}{2}$. Hence it follows from the definition of addition in cohomotopy groups that

$$\{wG\} + \{wG_1\} = \{w(F+F_1)\} = (F+F_1)^\#\{w\}.$$

But $\{wG\} = G^\#\{w\} = F^\#\{w\}$, $\{wG_1\} = F_1^\#\{w\}$, and (2.5) is proved.

Let (2.6) be true, let f, c be as in (2.7), and let F_0, F_1 be homotopies of the form (2.2) such that $F_0(x,0) = fx$, $F_0(x,1) = F_1(x,0)$, $d_{k+1}(F_1) = c$ and $F_0 X_I \subset \bar{Y}^k$, whence $d_{k+1}(F_0) = 0$. Then $d_{k+1}(F_0 + F_1) = c$, by (2.5). Therefore, if $f \simeq f'$ (in $\bar{Y}^k$), we may replace f by f' when proving (2.7). Hence we may assume, in consequence of (2.6), that $fX^m = y_0$. We also assume that $\bar{Y}^{k+1} \neq \bar{Y}^k$ since (2.6) is trivial if $\bar{Y}^{k+1} = \bar{Y}^k$.

Let $S \subset C_{k+1}(Y, B; \pi^k(X, A))$ be the set of all elements for which (2.7) is true with respect to every map $(X, A) \to (\bar{Y}^k, B)$. Then it follows from (2.5) that $S + S \subset S$. Let $e_1^{k+1}, \ldots, e_n^{k+1}$ be the $(k+1)$-cells in $Y - B$ and let $Y_i = \bar{Y}^k \cup e_i^{k+1}$. Then $C_{k+1}(Y, B; \pi^k(X, A))$ may be identified, by means of the injections, with the direct sum

$$C_{k+1}(Y_1, B; \pi^k(X, A)) + \ldots + C_{k+1}(Y_n, B; \pi^k(X, A)).$$

Hence, and since $S + S \subset S$, (2.7) will follow by induction on n when we have proved it for $n = 1$. So we assume that $\bar{Y}^{k+1} - \bar{Y}^k$ is a single (oriented) $(k+1)$-cell.

Let $u: (\bar{Y}^{k+1}, \bar{Y}^k) \to (S^{k+1}, z_0)$ be a map such that $u \mid \bar{Y}^{k+1} - \bar{Y}^k$ is of degree 1. Let θ, ϕ, etc., mean the same as in the paragraph containing (2.3) and let

$$v: (T, A_I) \to (S^k, z_0)$$

be a map such that $\phi\theta\{v\} = c\{u\} \in \pi^k(X, A)$. We assume, as we obviously may, that $v(X \times 0 \cup \bar{X}^{k-1} \times 1) = z_0$. Let

$$(X_I, T) \xrightarrow{w} (E^{k+1}, S^k) \xrightarrow{\psi} (\bar{Y}^{k+1}, \bar{Y}^k)$$

be, respectively, an extension of v such that† $wX_I^{k-1} = z_0$ and a map of degree 1 over $\bar{Y}^{k+1} - \bar{Y}^k$ such that $\psi z_0 = y_0$. Assume that $fX^m = y_0$ and define

$$G: (X_I, T) \to (\bar{Y}^{k+1}, \bar{Y}^k)$$

by $G(x, t) = fx$. Then $\psi w(X_I^{k-1} \cup A_I) = GX_I^m = y_0$ and

$$\dim(X_I - T) \leqslant k+m$$

since $\dim(X-A) \leqslant k+m-1$. Therefore a **D**-sum F' of ψw and G may be defined as in § 3. We have $\psi wA_I, GA_I \subset B$. Therefore F' determines a map

$$F: (X_I, T, A_I) \to (\bar{Y}^{k+1}, \bar{Y}^k, B).$$

Moreover $\rho\{u\psi w\} = \rho\Delta\{v\} = \phi\theta\{v\} = c\{u\}$. Therefore $(\psi w)^{\#} = \rho^{-1}c$ and $G^{\#} = 0$ since $GX_I \subset \bar{Y}^k$. Therefore it follows from (3.4) that $d_{k+1}(F) = c$.

Let X be imbedded in X_I so that $x = (x, 0)$ $(x \in X)$. Then $\psi wX = y_0$, since $vX = z_0$, and $Gx = fx$. Therefore it follows from (3.6) that F' determines a **D**-sum, $f': (X, A) \to (\bar{Y}^k, B)$, of y_0^* and f. We have $f' \simeq f$ (in $\bar{Y}^k$), by (3.3), and (2.7) is proved.

† $X_I^q = X^q \times I$, which contains the q-section of X_I.

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LOCAL DIMENSION OF NORMAL SPACES

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Introduction

LET $\dim X$ be the covering dimension of a space X and let $\operatorname{ind} X$ and $\operatorname{Ind} X$ be the dimensions defined inductively in terms of the boundaries of neighbourhoods of points and closed sets respectively. The local dimension $\operatorname{loc}\dim X$ is the least number n such that every point has a closed neighbourhood $\overline{U}$ with $\dim \overline{U} \leqslant n$. The local inductive dimension $\operatorname{loc}\operatorname{Ind} X$ is defined analogously, while $\operatorname{ind} X$ is already a local property.

The subset theorem, that $\dim A \leqslant \dim X$ for $A \subset X$, which was proved by E. Čech (**3**) for perfectly normal spaces is here extended to totally normal spaces. Čech's problem (**4**) of whether the subset theorem holds for completely normal Hausdorff spaces is reduced to the problem of whether the local dimension of a completely normal Hausdorff space is always equal to its dimension: that is, whether $\operatorname{loc}\dim X = \dim X$ for X completely normal and Hausdorff. But it follows from [3.7] below that a completely normal space X such that $\operatorname{loc}\dim X < \dim X$, if any such exists, must be neither paracompact nor the union of a sequence of closed paracompact sets nor the union of two paracompact sets one of which is closed. Thus most of the usual methods of constructing counter-examples are excluded.

However, a normal space M is constructed for which

$$\operatorname{loc}\dim M < \dim M.$$

Though this example is not completely normal, it is not clear that the lack of complete normality plays any significant role.

It is well known [(**6**) appendix] that a normal Hausdorff space may have a non-normal subspace of higher dimension. An example is given below of a normal Hausdorff space N with a normal subspace M such that $\dim N = \operatorname{Ind} N = 0$ but $\dim M = \operatorname{Ind} M = 1$.

The normal space M also has the property that $\operatorname{ind} M < \dim M$. Examples are known (**8**, **9**) of normal Hausdorff spaces X such that $\operatorname{ind} X > \dim X$. Thus for normal Hausdorff spaces there are the known relations $\operatorname{ind} X \leqslant \operatorname{Ind} X$ and $\dim X \leqslant \operatorname{Ind} X$ and no others.

A lemma which proves useful in several of the proofs below is the following (see [2.1]): *If a closed set A of a normal space X is at most n-dimensional, and if every closed set which does not meet A is at most n-dimensional, then* $\dim X \leqslant n$.

1. Definitions and elementary relations

A *covering* of a topological space X is a collection of open sets of X whose union is X. A covering β is called a *refinement* of a covering α if each member of β is contained in some member of α.

The *order* of a collection of subsets of a space X is -1 if all the subsets are empty; otherwise the *order* is the largest integer n such that some $n+1$ members of the collection have a non-empty intersection, or is ∞ if there is no such largest number.

The *dimension* of a space X, $\dim X$, is the least integer n such that every finite covering of X has a refinement of order not exceeding n, or the dimension is ∞ if there is no such integer.

[1.1] *For any space X, $\dim X \leqslant n$ if and only if, for every finite covering $\{U_1,\ldots, U_k\}$ of X, there is a covering $\{V_1,\ldots, V_k\}$ of order not exceeding n with each $V_i \subset U_i$.*

Proof. The condition is clearly sufficient, for $\{V_i\}$ is a refinement of $\{U_i\}$. To show necessity, let $\dim X \leqslant n$. Then the covering $\{U_i\}$ has some refinement β of order not exceeding n. Let each member of β be associated with one of the sets U_j containing it and let V_i be the union of the sets of β thus associated with U_i. Then V_i is open, $V_i \subset U_i$, and each point of X is in some member of β, and hence in some V_i. Each point p is in at most $n+1$ members of β, each of which is associated with a unique U_j, and hence p is in at most $n+1$ members of V_i. Thus V_i is a covering of order not exceeding n, as was to be shown.

The inductive dimensions $\operatorname{ind} X$ and $\operatorname{Ind} X$ are defined inductively as follows. If X is the empty set, $\operatorname{ind} X = \operatorname{Ind} X = -1$. If $\operatorname{ind} X \leqslant n-1$ has already been defined, $\operatorname{ind} X \leqslant n$ means that, for each point p and open set U with $p \in U$, there is an open set V with $p \in V \subset U$ for which $\operatorname{ind}(\overline{V}-V) \leqslant n-1$. Similarly, $\operatorname{Ind} X \leqslant n$ means that, for each closed set F and open set U with $F \subset U$, there is an open set V with $F \subset V \subset U$ for which $\operatorname{Ind}(\overline{V}-V) \leqslant n-1$. And $\operatorname{ind} X = \infty$ [$\operatorname{Ind} X = \infty$] means that there is no integer n for which $\operatorname{ind} X \leqslant n$ [$\operatorname{Ind} X \leqslant n$].

It is known [(**6**) appendix] that, for an arbitrary space X, $\dim X = 0$ if and only if $\operatorname{Ind} X = 0$. If A is any subset of a space X, then $\operatorname{ind} A \leqslant \operatorname{ind} X$ [(**6**) appendix]. If A is a closed subset of X, then

$$\dim A \leqslant \dim X \;[(\mathbf{3})\ \S 4] \quad \text{and} \quad \operatorname{Ind} A \leqslant \operatorname{Ind} X \;[(\mathbf{2})\ \S 16].$$

If p is a point of a space X, the *dimension of* X at p, $\dim_p X$, is defined as follows: $\dim_p X$ is the least integer n such that, for some open set U containing p, $\dim \overline{U} = n$ or, if there is no such integer, $\dim_p X = \infty$.

Similarly $\operatorname{ind}_p X$ [$\operatorname{Ind}_p X$] is defined to be the least integer n such that, for some open set U containing p, $\operatorname{ind} \overline{U} = n$ [$\operatorname{Ind} \overline{U} = n$] or, if there is no such integer, $\operatorname{ind}_p X = \infty$ [$\operatorname{Ind}_p X = \infty$]. It should be noted that the dimension at a point as defined here is entirely different from that defined by Menger (**10**) and Hurewicz and Wallman (**6**).

The *local dimension* of a space X, $\operatorname{loc\,dim} X$, is defined as follows. If X is empty, $\operatorname{loc\,dim} X = -1$. Otherwise, $\operatorname{loc\,dim} X$ is the least integer n such that, for every point $p \in X$, $\dim_p X \leqslant n$ or, if there is no such integer, $\operatorname{loc\,dim} X = \infty$.

The *local inductive dimensions*, $\operatorname{loc\,ind} X$ and $\operatorname{loc\,Ind} X$, are defined similarly. If X is empty, $\operatorname{loc\,ind} X = \operatorname{loc\,Ind} X = -1$. Otherwise, $\operatorname{loc\,ind} X$ [$\operatorname{loc\,Ind} X$] is the least integer n such that, for every point $p \in X$, $\operatorname{ind}_p X \leqslant n$ [$\operatorname{Ind}_p X \leqslant n$] or, if there is no such integer,

$$\operatorname{loc\,ind} X = \infty \quad [\operatorname{loc\,Ind} X = \infty].$$

Thus $\operatorname{loc\,dim} X$ [$\operatorname{loc\,ind} X$, $\operatorname{loc\,Ind} X$] is the least integer n such that there exists a covering $\{U_\lambda\}$ of X with each $\dim \overline{U}_\lambda \leqslant n$ [$\operatorname{ind} \overline{U}_\lambda \leqslant n$, $\operatorname{Ind} \overline{U}_\lambda \leqslant n$], or, if there is no such integer, $\operatorname{loc\,dim} X = \infty$ [$\operatorname{loc\,ind} X = \infty$, $\operatorname{loc\,Ind} X = \infty$].

[1.2] *For any point p of a space X,* $\dim_p X \leqslant n$ *if and only if each neighbourhood U of p contains a neighbourhood V of p with* $\dim \overline{V} \leqslant n$.

Proof. By the definition of $\dim_p X$, if there is any open set V with $p \in V$ and $\dim \overline{V} \leqslant n$, then $\dim_p X \leqslant n$. On the other hand, if

$$\dim_p X \leqslant n,$$

there exists a neighbourhood W of p with $\dim \overline{W} \leqslant n$. If $V = U \cap W$, then $p \in V \subset U$ and, since $\overline{V}$ is a closed subset of $\overline{W}$, $\dim \overline{V} \leqslant n$.

[1.3] *For any space X,* $\operatorname{loc\,dim} X \leqslant n$ *if and only if every covering of X has a refinement $\{U_\lambda\}$ with each* $\dim \overline{U}_\lambda \leqslant n$.

This follows immediately from [1.2]. Analogous propositions are clearly true of $\operatorname{loc\,ind} X$ and $\operatorname{loc\,Ind} X$.

[1.4] *For any space X,* $\operatorname{loc\,dim} X \leqslant \dim X$, $\operatorname{loc\,ind} X \leqslant \operatorname{ind} X$, *and* $\operatorname{loc\,Ind} X \leqslant \operatorname{Ind} X$.

Proof. If $\dim X \leqslant n$, then every point $p \in X$ has the neighbourhood X for which $\dim \bar{X} = \dim X \leqslant n$, and hence $\dim_p X \leqslant n$. Thus $\operatorname{loc} \dim X \leqslant n$. Similarly $\operatorname{ind} X \leqslant n$ implies $\operatorname{loc} \operatorname{ind} X \leqslant n$ and

$$\operatorname{Ind} X \leqslant n \quad \text{implies} \quad \operatorname{loc} \operatorname{Ind} X \leqslant n.$$

[1.5] *For any space* X, $\operatorname{loc} \operatorname{ind} X = \operatorname{ind} X$.

Proof. Let $\operatorname{loc} \operatorname{ind} X \leqslant n$ and let $p \in U \subset X$ with U open. Then there is some open set W with $p \in W$ and $\operatorname{ind} \bar{W} \leqslant n$. Hence $p \in U \cap W$ with $U \cap W$ open in $\bar{W}$. Hence, by the definition of $\operatorname{ind} \bar{W}$, there is a set V open in $\bar{W}$ with $p \in V \subset U \cap W$ and $\operatorname{ind} B \leqslant n-1$, where B is the boundary of V in $\bar{W}$. But, since V is open in $\bar{W}$, it is open in W and hence in X. Also the closure of V in $\bar{W}$ is its closure $\bar{V}$ in X and hence $B = \bar{V} - V$. Thus $p \in V \subset U$ with V open and $\operatorname{ind}(\bar{V} - V) \leqslant n-1$. Therefore

$$\operatorname{ind} X \leqslant n.$$

Thus $\operatorname{ind} X \leqslant \operatorname{loc} \operatorname{ind} X$, and so $\operatorname{loc} \operatorname{ind} X = \operatorname{ind} X$, as was to be shown.

Thus the inductive dimension $\operatorname{ind} X$ is already a local property of X. But $\dim X$ and $\operatorname{Ind} X$ are not in general local properties, as will be seen in § 6 below.

[1.6] *If* X *is a regular space*, $\operatorname{ind} X \leqslant \operatorname{loc} \operatorname{Ind} X$.

Proof. This is shown inductively. Clearly $\operatorname{loc} \operatorname{Ind} X = -1$ implies $\operatorname{ind} X = -1$. Assume it proved that

$$\operatorname{loc} \operatorname{Ind} X \leqslant n-1 \quad \text{implies} \quad \operatorname{ind} X \leqslant n-1.$$

Let $\operatorname{loc} \operatorname{Ind} X \leqslant n$ and let $p \in U \subset X$ with U open. Then, for some open set W, $p \in W$ and $\operatorname{Ind} \bar{W} \leqslant n$. Since X is regular, there is an open set G with $p \in G \subset \bar{G} \subset U \cap W$. Since $\operatorname{Ind} \bar{W} \leqslant n$, there is an open set V of $\bar{W}$ with $\bar{G} \subset V \subset U \cap W$ and $\operatorname{Ind} B \leqslant n-1$, where B is the boundary of V in $\bar{W}$. Since V is open in $\bar{W}$, it is open in W and hence in X. The closure of V in $\bar{W}$ is its closure $\bar{V}$ in X, and hence $B = \bar{V} - V$. Since $B \subset X$, B is regular and, since $\operatorname{Ind} B \leqslant n-1$, $\operatorname{loc} \operatorname{Ind} B \leqslant n-1$. Hence by the induction hypothesis $\operatorname{ind} B \leqslant n-1$. Thus $p \in V \subset U$ with V open and $\operatorname{ind}(\bar{V} - V) \leqslant n-1$. Hence $\operatorname{ind} X \leqslant n$, as was to be shown.

The above result, $\operatorname{ind} X \leqslant \operatorname{loc} \operatorname{Ind} X$, is in fact clearly true for any space in which the closure of each one-point set has no proper closed subset, and hence true for T_1 spaces as well as regular spaces.

It is known (**11**) that, if X is a normal space, $\dim X \leqslant \operatorname{Ind} X$. It follows that, for a normal space X, $\operatorname{loc} \dim X \leqslant \operatorname{loc} \operatorname{Ind} X$. For, if $\operatorname{loc} \operatorname{Ind} X \leqslant n$, then each point $p \in X$ has a neighbourhood U with

$\operatorname{Ind} \bar{U} \leqslant n$, and the closed set $\bar{U}$ of X is normal; hence $\dim \bar{U} \leqslant n$. Thus $\operatorname{loc}\dim X \leqslant n$.

Thus, using [1.4], [1.5], and [1.6], we have the following:

[1.7] *If X is a normal regular space, then*

$$\operatorname{ind} X \leqslant \operatorname{loc}\operatorname{Ind} X \leqslant \operatorname{Ind} X,$$

$$\operatorname{loc}\dim X \leqslant \operatorname{loc}\operatorname{Ind} X \leqslant \operatorname{Ind} X,$$

$$\operatorname{loc}\dim X \leqslant \dim X \leqslant \operatorname{Ind} X.$$

Since every normal Hausdorff space is regular, the inequalities of [1.7] hold in particular for normal Hausdorff spaces.

2. Properties of dimension

[2.1] *Let A be a closed set of a normal space X. If* $\dim A \leqslant n$ *and if* $\dim F \leqslant n$ *for every closed set F of X which does not meet A, then*

$$\dim X \leqslant n.$$

Proof. Let $\{U_1,..., U_k\}$ be a covering of X. Then, by (**3**) § 22, since X is normal and A is closed and $\dim A \leqslant n$, there exists a collection $\{V_i\}$ of open sets of X of order not exceeding n with each $V_i \subset U_i$ and $A \subset \bigcup_{i=1}^{k} V_i$. Let $V = \bigcup_{i=1}^{k} V_i$; then V is open and $A \subset V$.

Since X is normal, there exist open sets P and Q such that

$$X-V \subset P \subset \bar{P} \subset Q \subset \bar{Q} \subset X-A.$$

Since $\bar{Q}$ is closed and $\bar{Q} \cap A = 0$, $\dim \bar{Q} \leqslant n$. The sets $U_i \cap P$ together with the sets V_i form a collection of open sets of X covering the closed set $\bar{Q}$. Hence, by (**3**) § 22, $\bar{Q}$ is covered by a collection $\{G_i, H_j\}$ of open sets of X of order not exceeding n with $G_i \subset U_i \cap P$ and $H_j \subset V_j$.

Let $W_i = G_i \cup H_i \cup (V_i - \bar{P})$; then W_i is open in X and $W_i \subset U_i$. Each point of $\bar{P}$ is in at most $n+1$ of the sets $\{G_i, H_j\}$ and in none of the sets $V_i - \bar{P}$; hence it is in at most $n+1$ of the sets W_i. Each point of $X-\bar{P}$ is in none of the sets G_i and in at most $n+1$ of the sets V_i and hence, since $H_i \subset V_i$, it is in at most $n+1$ of the sets W_i. Thus $\{W_i\}$ is of order not exceeding n. Each point of $\bar{P}$ is contained in $\bar{Q}$ and hence in some G_i or H_j; hence it is in some W_i. And each point of $X-\bar{P}$ is in some $V_i - \bar{P}$ and hence in some W_i. Thus $\{W_i\}$ is a covering of X. Thus $\{W_i\}$ is a refinement of $\{U_i\}$ of order not exceeding n. Hence $\dim X \leqslant n$, as was to be shown.

[2.2] *If a normal space X is the union of two sets A and B with A closed and* $\dim A \leqslant n$ *and* $\dim B \leqslant n$, *then* $\dim X \leqslant n$.

Proof. If F is a closed set of X which does not meet A, then F is a closed subset of B and $\dim F \leqslant \dim B \leqslant n$. Hence, by [2.1], $\dim X \leqslant n$, as was to be shown.

[2.3] *If A is a closed set of a normal space X, then*

$$\dim X \leqslant \max(\dim A, \dim(X-A)).$$

Proof. This follows immediately from [2.2] on setting $B = X-A$.

A normal space X is called *totally normal* (**5**) if each open subspace Y of X has a locally finite covering by open subsets each of which is an F_σ set of X. As will be shown in [2.6] below, the inequality in [2.3] becomes equality if X is totally normal.

[2.4] *Each totally normal space is regular.*

Proof. If X is totally normal, let $p \in Y \subset X$ with Y open. Then p is in some open F_σ set U contained in Y and, since U is an F_σ set, there is a closed set F with $p \in F \subset U$. Since X is normal, there is an open set V with $F \subset V \subset \overline{V} \subset U$; then $p \in V \subset \overline{V} \subset Y$. Thus X is regular.

[2.5] *Let a space X be the union of disjoint sets A each of which is open and closed in X. If each* $\dim A_\lambda \leqslant n$, *then* $\dim X \leqslant n$.

Proof. Let $\{U_1,..., U_k\}$ be any finite covering of X. Then

$$\{U_1 \cap A_\lambda,..., U_k \cap A_\lambda\}$$

is a covering of A_λ and $\dim A_\lambda \leqslant n$; hence there is a covering $\{V_{\lambda i}\}$ of A_λ of order not exceeding n with $V_{\lambda i} \subset U_i \cap A_\lambda$. Let $V_i = \bigcup_\lambda V_{\lambda i}$; then $\{V_i\}$ is a covering of X of order not exceeding n and $V_i \subset U_i$. Therefore

$$\dim X \leqslant n,$$

as was to be shown.

[2.6] *If Y is an open set of a totally normal space X, then* $\dim Y \leqslant \dim X$.

Proof. Let $\dim X \leqslant n$; it is sufficient to show that $\dim Y \leqslant n$. Since Y is an open set of a totally normal space X, we know [(**5**) proposition 4.3] that for each $i = 1, 2,...$ there is a collection $\{W_{i\lambda}\}$, locally finite in Y, of disjoint open sets and a corresponding collection $\{F_{i\lambda}\}$ of closed sets of X such that $F_{i\lambda} \subset W_{i\lambda} \subset Y$ and $\bigcup_{i=1}^{\infty} \bigcup_\lambda F_{i\lambda} = Y$. Since $F_{i\lambda}$ is closed in X, $\dim F_{i\lambda} \leqslant \dim X \leqslant n$. Let $F_i = \bigcup_\lambda F_{i\lambda}$; then since, for fixed i, $\{F_{i\lambda}\}$ is locally finite in Y, F_i is closed in Y. Likewise $\bigcup_{\mu \neq \lambda} F_{i\mu}$ is closed in Y, and hence $F_{i\lambda} = F_i - \bigcup_{\mu \neq \lambda} F_{i\mu}$ is open in F_i. Therefore, by [2.5], $\dim F_i \leqslant n$. Since X is totally normal, it is completely normal [(**5**) proposition 4.6],

and hence Y is normal. By the sum theorem [(**3**) § 23], since $Y = \bigcup_{i=1}^{\infty} F_i$ with F_i closed in Y and $\dim F_i \leqslant n$, therefore $\dim Y \leqslant n$, as was to be shown.

[2.7] *If, for each open set Y of a space X, $\dim Y \leqslant n$, then, for each set A of X, $\dim A \leqslant n$.*

Proof. Let $\{G_1,..., G_k\}$ be a covering of A. Each G_i is open in A and hence there is an open set U_i of X such that $G_i = U_i \cap A$. Let $Y = \bigcup_{i=1}^{k} U_i$; then $A \subset Y$ and Y is open in X, and hence $\dim Y \leqslant n$. Hence there is a covering $\{V_i\}$ of Y of order not exceeding n with each $V_i \subset U_i$. Then $\{V_i \cap A\}$ is a covering of A of order not exceeding n and

$$V_i \cap A \subset U_i \cap A = G_i.$$

Hence $\dim A \leqslant n$ as was to be shown.

The subset theorem, which was proved by Čech [(**3**) § 28] for perfectly normal spaces can be extended to totally normal spaces as follows.

[2.8] *If A is a set in a totally normal space X, then* $\dim A \leqslant \dim X$.

Proof. This follows immediately from [2.6] and [2.7].

It is also true [(**5**) Theorem 2] that, if A is any subset of a totally normal space X, then $\operatorname{Ind} A \leqslant \operatorname{Ind} X$.

3. Relation of local dimension to dimension

For any space X one has the trivial relation $\operatorname{loc\,dim} X \leqslant \dim X$, and an example is given below of a normal Hausdorff space M with

$$\operatorname{loc\,dim} M < \dim M.$$

But, as is now to be shown, in [3.3], [3.5], and [3.6], there is a wide class of normal spaces for which $\operatorname{loc\,dim} X = \dim X$.

[3.1] *If A is a closed set of a space X*, $\operatorname{loc\,dim} A \leqslant \operatorname{loc\,dim} X$.

Proof. Let $\operatorname{loc\,dim} X \leqslant n$ and let x be a point of A. Then there is a neighbourhood U of x in X with $\dim \overline{U} \leqslant n$. Then $U \cap A$ is a neighbourhood of x in A and the closure of $U \cap A$ in A is a closed subset of $\overline{U}$ and hence has dimension not exceeding n. Hence $\operatorname{loc\,dim} A \leqslant n$. Thus

$$\operatorname{loc\,dim} A \leqslant \operatorname{loc\,dim} X,$$

as was to be shown.

[3.2] *If $\{U_\lambda\}$ is any covering of a paracompact normal space X, then X is the union of a sequence of closed sets $\{H_i\}$ each of which is the union of a collection $\{H_{i\mu}\}$ of disjoint sets with each $H_{i\mu}$ open and closed in H_i and with each $H_{i\mu}$ contained in some U_λ.*

Proof. Since X is paracompact, $\{U_\lambda\}$ has a locally finite refinement $\{V_\mu\}$. Since X is normal, $\{V_\mu\}$ can be shrunk [(**7**) (I, 33, 4)] to a covering $\{W_\mu\}$ with $\overline{W}_\mu \subset V_\mu$. There exist real continuous functions f_μ $(0 \leqslant f_\mu(x) \leqslant 1)$ such that $f_\mu(x) = 0$ if $x \in X - V_\mu$ and $f_\mu(x) = 1$ if $x \in \overline{W}_\mu$. Let $F_{i\mu}$ be the set of points x for which $f_\mu(x) \geqslant 1/i$ and let G_μ be the set of points for which $f_\mu(x) > 0$.

Let the indices μ be well ordered. Let $H_{i\mu}$ be the set $F_{i\mu} - \bigcup_{\nu<\mu} G_\nu$ and let $H_i = \bigcup_\mu H_{i\mu}$. The sets $F_{i\mu}$ are closed, G_ν is open, and each $H_{i\mu}$ is closed. If $\nu < \mu$, $H_{i\nu} \subset G_\nu$ and $H_{i\nu} \cap H_{i\mu} = 0$. Thus, for each i, the sets $H_{i\mu}$ are disjoint. Since $H_{i\mu} \subset V_\mu$, the collection $\{H_{i\mu}\}$ for fixed i is locally finite; hence H_i is closed and $\bigcup_{\nu \neq \mu} H_{i\nu}$ is closed, whence $H_{i\mu}$ is open in H_i. Thus H_i is the union of disjoint sets $H_{i\mu}$, each open and closed in H_i, and $H_{i\mu} \subset V_\mu$, which is contained in some U_λ.

Each point $x \in X$ is in some W_μ and hence in some G_μ. Then, if we take the first μ for which $x \in G_\mu$, $x \in F_{i\mu}$ for some i while x is in none of the sets G_ν with $\nu < \mu$. Hence $x \in H_{i\mu}$ for some i, and therefore $x \in H_i$. Thus $X = \bigcup_{i=1}^{\infty} H_i$, as was to be shown.

[3.3] *If X is a paracompact normal space, then* $\operatorname{loc\,dim} X = \dim X$.

Proof. Let $\operatorname{loc\,dim} X \leqslant n$. Then each point $x \in X$ is in some open set U_x such that $\dim \overline{U}_x \leqslant n$. Then, by [3.2], X is the union of a sequence of closed sets $\{H_i\}$ each of which is the union of a collection $\{H_{i\mu}\}$ of disjoint sets with each $H_{i\mu}$ contained in some U_x. Then $H_{i\mu}$ is a closed subset of some $\overline{U}_x$, and hence $\dim H_{i\mu} \leqslant n$. Hence, by [2.5], $\dim H_i \leqslant n$. Hence, by the sum theorem [(**3**) § 23], since $X = \bigcup_{i=1}^{\infty} H_i$ with H_i closed and $\dim H_i \leqslant n$, we have $\dim X \leqslant n$. Thus $\dim X \leqslant \operatorname{loc\,dim} X$, and hence $\operatorname{loc\,dim} X = \dim X$, as was to be shown.

[3.4] *If X is a paracompact totally normal space, then*

$$\operatorname{loc\,Ind} X = \operatorname{Ind} X.$$

Proof. Let $\operatorname{loc\,Ind} X \leqslant n$. Then each point $x \in X$ is in some open set U_x such that $\operatorname{Ind} \overline{U}_x \leqslant n$. Then, by [3.2], X is the union of a sequence of closed sets $\{H_i\}$ each of which is the union of a collection $\{H_{i\mu}\}$ of disjoint sets with each $H_{i\mu}$ open and closed in H_i and with each $H_{i\mu}$ contained in some U_x. Then $H_{i\mu}$ is a closed subset of some $\overline{U}_x$, and hence $\operatorname{Ind} H_{i\mu} \leqslant n$. Hence, by (**5**) proposition 5.1, $\operatorname{Ind} H_i \leqslant n$. By the sum theorem [(**5**) Theorem 4] for the inductive dimension of totally normal

spaces, since $X = \bigcup_{i=1}^{\infty} H_i$ with H_i closed and $\operatorname{Ind} H_i \leqslant n$, therefore $\operatorname{Ind} X \leqslant n$. Thus $\operatorname{Ind} X \leqslant \operatorname{loc} \operatorname{Ind} X$, and hence $\operatorname{loc} \operatorname{Ind} X = \operatorname{Ind} X$, as was to be shown.

[3.5] *If a normal space X is the union of a sequence $\{A_i\}$ of closed paracompact subsets, then* $\operatorname{loc} \dim X = \dim X$.

Proof. Let $\operatorname{loc} \dim X \leqslant n$. Then, by [3.1], since A_i is closed,

$$\operatorname{loc} \dim A_i \leqslant n.$$

Since X is normal, the closed set A_i is normal. Hence, by [3.3], since A_i is paracompact, $\dim A_i \leqslant n$. By the sum theorem [(**3**) § 23], since $X = \bigcup_{i=1}^{\infty} A_i$ with A_i closed and $\dim A_i \leqslant n$, therefore $\dim X \leqslant n$. Thus $\dim X \leqslant \operatorname{loc} \dim X$, and hence $\operatorname{loc} \dim X = \dim X$.

[3.6] *If a normal space X is the union of two paracompact sets A and B with A closed in X, then* $\operatorname{loc} \dim X = \dim X$.

Proof. Let $\operatorname{loc} \dim X \leqslant n$. Then, by [3.1], since A is closed,

$$\operatorname{loc} \dim A \leqslant n.$$

Since X is normal, the closed set A is normal. Hence, by [3.3], since A is paracompact, $\dim A \leqslant n$. Let F be any closed set of X which does not meet A; then F is normal and $\operatorname{loc} \dim F \leqslant n$. Since F is a closed subset of B, F is paracompact and hence, by [3.3], $\dim F \leqslant n$. Hence, by [2.1], $\dim X \leqslant n$. Thus $\dim X \leqslant \operatorname{loc} \dim X$, and hence

$$\operatorname{loc} \dim X = \dim X.$$

[3.7] *Let X be an n-dimensional normal space. If X is paracompact, or the union of two paracompact sets one of which is closed, or the union of a sequence of closed paracompact sets, then the set of points of X at which X is n-dimensional is an n-dimensional closed set of X.*

Proof. Let D be the set of points of X at which X is n-dimensional and let $p \in X - D$. Since $\operatorname{loc} \dim X \leqslant \dim X = n$, $\dim_p X < n$ and, for some neighbourhood U of p, $\dim \overline{U} \leqslant n-1$. Then, for each point $x \in U$, $\dim_x X \leqslant n-1$, and hence $x \in X - D$. Thus $X - D$ is open and D is closed.

Let F be any closed set of X which does not meet D. Then each point x of F has a neighbourhood U in X such that $\dim \overline{U} \leqslant n-1$. Then $U \cap F$ is a neighbourhood of x in F and its closure in F is a closed subset of $\overline{U}$ and hence has dimension not exceeding $n-1$. Thus

$\operatorname{loc}\dim F \leqslant n-1$. If X is paracompact, then the closed set F is paracompact. If $X = A \cup B$ with A closed and both A and B paracompact, then

$$F = (A \cap F) \cup (B \cap F)$$

with $A \cap F$ closed and, since $A \cap F$ is closed in A and $B \cap F$ is closed in B, $A \cap F$ and $B \cap F$ are paracompact. If $X = \bigcup_{i=1}^{\infty} A_i$ with each A_i closed and paracompact, then $F = \bigcup_{i=1}^{\infty} A_i \cap F$ and each $A_i \cap F$ is closed and paracompact. The closed set F of X is normal; hence, by [3.3] or [3.5] or [3.6], $\dim F \leqslant n-1$.

By [2.1], if the dimension of D were $\leqslant n-1$, then we should have $\dim X \leqslant n-1$, which is absurd. Thus $\dim D \geqslant n$ and, since D is closed in X, $\dim D \leqslant \dim X = n$. Hence $\dim D = n$, as was to be shown.

In reference to the hypotheses of [3.5], [3.6], and [3.7], note that Bing's example H [see (**1**)] is a normal space which is the union of a countable number of discrete and hence paracompact (even metrizable) closed subsets but it is not paracompact. Either of his examples G or H is a non-paracompact normal space which is the union of two discrete and hence paracompact subsets, one of which is closed.

Nor does the normality of X follow from the other properties. For example, let R be a non-countable set of points, one of which is called r_0. The open sets of R are the sets not containing r_0 and the sets containing all but a finite number of points of R. Then R is a normal space: in fact it is a compact Hausdorff space. Let S have a countably infinite set of points and let its open sets be sets not containing a special point s_0 and sets containing all but a finite number of points. Let X be the subset of $R \times S$ formed by removing the point (r_0, s_0). Then X is a Hausdorff space.

If
$$A = ((R \times s_0) \cup (r_0 \times S)) \cap X,$$

then A is closed in X and both A and $X-A$ are discrete and hence paracompact. Also the sets $R \times s$ for $s \neq s_0$ are compact and $(R \times s_0) \cap X$ is discrete; thus X is the union of a countable collection of closed paracompact subsets. But X is neither normal nor paracompact.

4. Properties of local dimension

As has been shown in [3.1] above, the closed-subset theorem holds for the local dimension of arbitrary spaces. I now show that the open-subset theorem holds for the local dimension, though not necessarily for the dimension (see § 7 below), of regular spaces. And the subset

theorem of local dimension holds for totally normal spaces. The finite-sum theorem, but not the countable-sum theorem (see § 7 below), holds for the local dimension of normal spaces.

[4.1] *If Y is an open set of a regular space X, then* $\operatorname{loc}\dim Y \leqslant \operatorname{loc}\dim X$ *and* $\operatorname{loc}\operatorname{Ind} Y \leqslant \operatorname{loc}\operatorname{Ind} X$.

Proof. Let $\operatorname{loc}\dim X \leqslant n$ and let x be a point of Y. There is a neighbourhood U of x in X such that $\dim \overline{U} \leqslant n$. Since X is regular, there is an open set V containing x whose closure $\overline{V}$ is contained in the open set $U \cap Y$. Then V is a neighbourhood of x in Y, $\overline{V}$ is the closure of V in Y, and, since $\overline{V}$ is a closed subset of $\overline{U}$, $\dim \overline{V} \leqslant n$. Thus $\operatorname{loc}\dim Y \leqslant n$. Hence $\operatorname{loc}\dim Y \leqslant \operatorname{loc}\dim X$. The proof that $\operatorname{loc}\operatorname{Ind} Y \leqslant \operatorname{loc}\operatorname{Ind} X$ is similar and is omitted.

[4.2] *If A is a subset of a totally normal space X, then*

$$\operatorname{loc}\dim A \leqslant \operatorname{loc}\dim X \quad \textit{and} \quad \operatorname{loc}\operatorname{Ind} A \leqslant \operatorname{loc}\operatorname{Ind} X.$$

Proof. Let $\operatorname{loc}\dim X \leqslant n$ and let x be a point of A. There is a neighbourhood U of x in X such that $\dim \overline{U} \leqslant n$. Then $U \cap A$ is a neighbourhood of x in A whose closure in A is a subset of the totally normal space $\overline{U}$ and hence, by [2.8], has dimension not exceeding n. Thus $\operatorname{loc}\dim A \leqslant n$. Hence $\operatorname{loc}\dim A \leqslant \operatorname{loc}\dim X$. The proof that $\operatorname{loc}\operatorname{Ind} A \leqslant \operatorname{loc}\operatorname{Ind} X$ is similar but uses (**5**), Theorem 2, instead of [2.8].

[4.3] *If a normal space X is the union of two closed sets A and B and if* $\operatorname{loc}\dim A \leqslant n$ *and* $\operatorname{loc}\dim B \leqslant n$, *then* $\operatorname{loc}\dim X \leqslant n$.

Proof. If $x \in X - A$, then $x \in B$ and there is an open set $U \cap B$ of B, where U is open in X, such that $\dim \overline{U \cap B} \leqslant n$. If $W = U \cap (X - A)$, then W is open in X, $x \in W$, and, since $\overline{W} \subset \overline{U \cap B}$, $\dim \overline{W} \leqslant n$. Similarly, if $x \in X - B$, there is an open set of X containing x whose closure has dimension not exceeding n. If $x \in A \cap B$, then there exist open sets U and V containing x such that $\dim \overline{U \cap A} \leqslant n$ and $\dim \overline{V \cap B} \leqslant n$. Let

$$W = X - (A - U) - (B - V);$$

then W is open, $x \in W$, and $W \subset (U \cap A) \cup (V \cap B)$. Then

$$\dim \overline{W} \leqslant \dim(\overline{U \cap A} \cup \overline{V \cap B}) \leqslant \max(\dim \overline{U \cap A}, \dim \overline{V \cap B}) \leqslant n$$

since the sum theorem [(**3**) § 23] holds in the normal space $\overline{W}$. Thus $\operatorname{loc}\dim X \leqslant n$, as was to be shown.

5. The subset theorem and local dimension

In this section the problem of whether the subset theorem of dimension holds for all completely normal regular spaces is reduced to the

apparently simpler problem of whether the local dimension of every completely normal regular space is equal to its dimension.

If one drops the condition of regularity, there are trivial counter-examples to the subset theorem. For example, let I be a line segment and let X be the space consisting of I together with one additional point x_0, the open sets of X being the open sets of I and the whole space X. Then X is completely normal but not regular, and $\dim X = 0$ while $\dim I = 1$.

[5.1] *If X is any normal regular space, there is a normal regular space X^* containing X as an open subset such that* $\dim X^* \leqslant \operatorname{loc}\dim X$. *If X is a Hausdorff space or a completely normal space, so is X^*.*

Proof. If X is empty, let $X^* = X$. Otherwise the points of the space X^* are the points of X together with one new point x_0. A set U of X^* is to be open if either (i) $U \subset X$ and U is open in X or (ii) $x_0 \in U$ and $X^* - U$ is a closed set of X which is contained in an open set V of X such that $\dim \overline{V} \leqslant \operatorname{loc}\dim X$.

It is clear that the empty set is an open set of type (i) and the whole space X^* is an open set of type (ii). The intersection of two open sets, one of which is of type (i), is an open set of type (i). If U_1 and U_2 are open sets of type (ii), then $x_0 \in U_1 \cap U_2$ and

$$X^* - (U_1 \cap U_2) = (X^* - U_1) \cup (X^* - U_2),$$

which is the union of two closed sets and hence is a closed set of X.

If $X^* - U_1 \subset V_1$ and $X^* - U_2 \subset V_2$ with V_1 and V_2 open in X and

$$\dim \overline{V}_1 \leqslant \operatorname{loc}\dim X \quad \text{and} \quad \dim \overline{V}_2 \leqslant \operatorname{loc}\dim X,$$

then $X^* - (U_1 \cap U_2) \subset V_1 \cup V_2$ and $\overline{V_1 \cup V_2} = \overline{V}_1 \cup \overline{V}_2$ is a closed set of X and hence is normal. Therefore, by the sum theorem,

$$\dim(\overline{V}_1 \cup \overline{V}_2) = \max(\dim \overline{V}_1, \dim \overline{V}_2) \leqslant \operatorname{loc}\dim X.$$

Thus the intersection $U_1 \cap U_2$ is an open set of type (ii).

The union of any collection of open sets of type (i) is again an open set of type (i). If the collection contains an open set U_1 of type (ii), then the union U contains x_0, and $U - (x_0)$ is a union of open sets of X, and hence is open in X. Therefore $X^* - U$ is closed in X and, if

$$X^* - U_1 \subset V_1$$

with V_1 open in X and $\dim \overline{V}_1 \leqslant \operatorname{loc}\dim X$, then $X^* - U$ is also contained in V_1. Therefore U is an open set of type (ii). Thus X^* is a topological space, and clearly X is a subspace. The set X is an open set of type (i) in X^*.

The space X^* is normal. For, if E and F are disjoint closed sets of X^*, at least one of them, say F, does not contain x_0. Then X^*-F is an open set of X^* containing x_0; hence it is an open set of type (ii). Therefore there is an open set V of X with $F \subset V$ and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Then $V \cap (X^*-E)$ is an open set of X containing the closed set F. Since X is normal, there exists an open set W of X such that

$$F \subset W \subset \bar{W} \subset V \cap (X^*-E).$$

Let $U = X^*-\bar{W}$; then $x_0 \in U$, the set $X^*-U = \bar{W}$ is closed in X, $\bar{W} \subset V$ with V open in X, and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Therefore U is an open set of type (ii) while the set W is open of type (i) in X^*. We have $F \subset W$ and, since $\bar{W} \subset X^*-E$,

$$E \subset X^*-\bar{W} = U, \qquad U \cap W = 0.$$

Therefore X^* is normal.

The space X^* is regular. For, if $x \in U \subset X^*$ with U open in X^*, then either $x = x_0$ and U is open of type (ii) or $x \neq x_0$ and $x \in U \cap X$, which is open of type (i) in X^*. If $x = x_0 \in U$, then, since X is open in X^*, (x_0) is closed and, by the normality of X^*, there is an open set W with $x_0 \in W \subset \bar{W} \subset U$, where $\bar{W}$ is the closure of W in X^*. If $x \neq x_0$, then, by the definition of local dimension, there is some neighbourhood V of x in X such that $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Since X is regular, there is an open set W of X with $x \in W \subset \bar{W} \subset V \cap U$. Since $\bar{W}$ is closed in X and $\bar{W} \subset V$ with V open and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$, therefore $X^*-\bar{W}$ is open of type (ii) and $\bar{W}$ is closed in X^*. Thus $x \in W \subset \bar{W} \subset U$ with W open and $\bar{W}$ closed in X^*. Therefore X^* is regular.

The set (x_0) is closed in X^* and $\dim(x_0) = 0 \leqslant \operatorname{loc}\dim X$ since X is non-empty. Let F be any closed set of X^* which does not meet (x_0). Then X^*-F is an open set of type (ii) and hence F is closed in X and $F \subset V$ for some open set V of X with $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Therefore $\dim F \leqslant \operatorname{loc}\dim X$. Hence, by [2.1], since X^* is normal,

$$\dim X^* \leqslant \operatorname{loc}\dim X.$$

Let X be a Hausdorff space. Then, if $x \in X$, (x) is a closed set of X. And, since $\dim_x X \leqslant \operatorname{loc}\dim X$, there is an open set V of X with $x \in V$ and $\dim \bar{V} \leqslant \operatorname{loc}\dim X$. Hence $X^*-(x)$ is an open set of type (ii) and (x) is a closed set of X^*. Since X is open in X^*, (x_0) is a closed set of X^*. Thus all one-point sets of X^* are closed. Hence, since X^* is normal, X^* is a Hausdorff space.

Let X be completely normal. If U is an open set of type (ii) in X^*, then X^*-U is closed in X and $X^*-U \subset V$ with V open in X and

$\dim \overline{V} \leqslant \operatorname{loc}\dim X$. Since X is normal, there is an open set W of X with $X^* - U \subset W \subset \overline{W} \subset V$. Then $X^* - \overline{W}$ is open of type (ii) and $\overline{W}$ is closed in X^*. Since $X^* - W$ is closed in the normal space X^*, $X^* - W$ is normal. Since $\overline{W} \cap U$ is a subset of the completely normal space X, $\overline{W} \cap U$ is normal. Then U is the union of two relatively closed normal subsets $X^* - W$ and $\overline{W} \cap U$; hence [(**12**) 186, lemma] U is normal.

And, if U is an open set of type (i), then $U \subset X$ and hence U is normal. Thus every open set of X^* is normal and hence [(**5**) proposition 1.1] X^* is completely normal. This completes the proof of [5.1].

[5.2] *If X is a normal regular space such that* $\dim X > \operatorname{loc}\dim X$, *then there is a normal regular space X^* containing X as an open subset such that* $\dim X > \dim X^*$. *If X is a Hausdorff space or a completely normal space, so is X^*.*

Proof. By [5.1] we have $\dim X^* \leqslant \operatorname{loc}\dim X$. Then, since

$$\dim X > \operatorname{loc}\dim X,$$

therefore $\dim X > \dim X^*$. The remaining conclusions follow from [5.1].

[5.3] *If X is a completely normal regular space with a subset A such that* $\dim A > \dim X$, *then X has an open subset Y such that*

$$\dim Y > \operatorname{loc}\dim Y.$$

Proof. Since $\dim X < \dim A$, $\dim X$ is finite. Since $\dim A > \dim X$, there is a covering $\{G_1,..., G_k\}$ of A which has no refinement of order not exceeding $\dim X$. Let $G_i = A \cap U_i$ with U_i open in X, and let $Y = \bigcup_{i=1}^{k} U_i$. Then Y is open in X and the covering $\{U_i\}$ of Y has no refinement of order not exceeding $\dim X$. Therefore $\dim Y > \dim X$. But, by [4.1],

$$\operatorname{loc}\dim Y \leqslant \operatorname{loc}\dim X \leqslant \dim X.$$

Therefore $\dim Y > \operatorname{loc}\dim Y$, as was to be shown.

6. An example

We now construct an example of a normal Hausdorff space M such that $\operatorname{loc}\dim M < \dim M$. Let T be the space consisting of the ordinal numbers less than ω_1 with the usual order topology [(**6**) appendix]. For each $\alpha \in T$, let

$$T_\alpha = \{\beta : \beta \leqslant \alpha\}, \qquad T'_\alpha = \{\beta : \beta \in T, \beta > \alpha\}.$$

Then, for each α, T_α and T'_α are disjoint closed sets of T whose union is T.

[6.1] *If $\{U_i\}$ is a countable (or finite) covering of T, then, for some integer j and some $\alpha \in T$, $T'_\alpha \subset U_j$.*

Proof. Assume on the contrary that for each α and j there is some $\beta \in T$ with $\beta > \alpha$ such that the interval $(\alpha, \beta) = \{\xi: \alpha < \xi < \beta\}$ is not contained in U_j. Let the least such β be $\beta_j(\alpha)$. Let $\gamma(\alpha)$ be the least upper bound of the sequence of ordinal numbers $\beta_j(\alpha)$. Then

$$\alpha < \gamma(\alpha) < \omega_1 \quad \text{and} \quad \gamma(\alpha) \in T.$$

Let $\alpha_1 = \gamma(0)$, $\alpha_2 = \gamma(\alpha_1), \ldots, \alpha_{r+1} = \gamma(\alpha_r), \ldots$. Then the sequence $\{\alpha_r\}$ has a least upper bound δ in T and $\delta > a_r$ since $\alpha_{r+1} > \alpha_r$. But δ is in some set U_j of the covering and is in some interval (α, β) contained in U_j. Then $\alpha < \delta$, and hence, for some r, $\alpha < \alpha_r$ and

$$\alpha_{r+1} = \gamma(\alpha_r) \geqslant \beta > \delta,$$

which is absurd. This completes the proof.

Let I be the space of real numbers, $0 \leqslant p \leqslant 1$, and let the numbers $p \in I$ be divided into congruence classes *modulo* the rational numbers. There are $\mathbf{c}$ such classes and $\mathbf{c} \geqslant \aleph_1$. Let $\aleph_1$ of these classes Q_α be chosen and indexed by the ordinal numbers $\alpha \in T$.

Example M. Let M be the subspace of the product space $T \times I$ consisting of those pairs (α, p) for which $p \notin \bigcup_{\beta \geqslant \alpha} Q_\beta$.

We define a *special* covering σ of M as follows. For some irreducible covering $\{W_1, \ldots, W_k\}$ of I by intervals open in I and for some $\alpha \in T$, σ consists of the covering of $M'_\alpha = (T'_\alpha \times I) \cap M$ by the sets $(T'_\alpha \times W_i) \cap M$, together with a covering of $M_\alpha = (T_\alpha \times I) \cap M$ by a finite number of disjoint open (and closed) sets. We may assume that $0 \in W_1$, $1 \in W_k$, and, for $i = 1, \ldots, k-1$, $W_i \cap W_{i+1}$ is not empty.

[6.2] *For each finite covering $\{U_i\}$ of M and $p \in I$ there is a neighbourhood (open interval) W of p in I and an $\alpha \in T$ such that, for some U_j of the covering,*

$$(T'_\alpha \times W) \cap M \subset U_j.$$

Proof. For each $p \in I$ there exists some $\beta \in T$ such that $T'_\beta \times p \subset M$; if $p \in Q_\alpha$, it is sufficient to take $\beta > \alpha$ while, if p is in no Q_α, one may take $\beta = 0$. Let $W_n(p)$ be the n^{-1}-neighbourhood of p in I and let $V(j, n)$ be the set of points α of T'_β such that, for some $\gamma < \alpha$,

$$((\gamma, \alpha+1) \times W_n(p)) \cap M \subset U_j.$$

Clearly $V(j, n)$ is an open set in T'_β.

For each $\alpha \in T'_\beta$, $(\alpha, p) \in M$, and hence $(\alpha, p) \in U_j$ for some U_j of the covering. There is an open set G_j of $T \times I$ such that $U_j = G_j \cap M$.

Then $(\alpha, p) \in U_j$, and hence there is some product neighbourhood $(\gamma, \alpha+1) \times W_n(p)$ of (α, p) contained in G_j. Then

$$((\gamma, \alpha+1) \times W_n(p)) \cap M \subset U_j,$$

and hence $\alpha \in V(j, n)$. Thus $\{V(j, n)\}$ is a covering of T'_β, and, since j and n take a finite and countable number of values respectively, the covering is countable. Adding the open set T_β, we get a countable covering of T.

By [6.1] there exist $j(p)$, $n(p)$, and $\alpha(p)$ such that $T'_{\alpha(p)} \subset V(j(p), n(p))$, and hence $(T'_{\alpha(p)} \times W_{n(p)}(p)) \cap M \subset U_j$. Thus it is sufficient to take $\alpha = \alpha(p)$ and $W = W_{n(p)}(p)$.

[6.3] *Every finite covering $\{U_i\}$ of M has a special refinement.*

Proof. By [6.2], for each $p \in I$ there is a neighbourhood $W(p)$ and an element $\alpha(p)$ of T such that $(T'_{\alpha(p)} \times W(p)) \cap M \subset U_j$ for some j. Since I is compact, the covering $\{W(p)\}$ of I contains an irreducible finite covering $\{W_k\}$ with $W_k = W(p_k)$. Let α be the greatest of the corresponding ordinal numbers $\alpha(p_k)$. Then for each W_k there is some U_j such that

$$(T'_\alpha \times W_k) \cap M \subset U_j.$$

The space $M_\alpha = (T_\alpha \times I) \cap M$ is a subspace of $T_\alpha \times (I - Q_\alpha)$, which is a zero-dimensional separable metrizable space. Hence the covering of M_α by the sets $U_i \cap M_\alpha$ has a finite refinement which is a covering by disjoint open sets. This, together with the collection of sets $(T'_\alpha \times W_k) \cap M$ which cover $M'_\alpha = (T'_\alpha \times I) \cap M$, forms the required special covering of M. This completes the proof.

A covering $\{U_i\}$ of a space X is called *shrinkable* if there is a covering $\{V_i\}$ of X such that $\overline{V}_i \subset U_i$. A space X is normal if and only if each finite covering of X is shrinkable [(**7**) 26], or, equivalently, if and only if each finite covering of X has a shrinkable finite refinement. In particular the covering $\{W_i\}$ of I is shrinkable, and hence each special covering of M is shrinkable. Therefore, since every finite covering has a special refinement, M is a normal space.

Since T and I are Hausdorff spaces, the subspace M of $T \times I$ is a Hausdorff space. Hence, since M is normal, it is a regular space.

It can easily be shown that M is countably paracompact but is not paracompact, not countably compact, and not completely normal.

[6.4] *For the normal Hausdorff space M we have*

$$\operatorname{ind} M = \operatorname{loc\,dim} M = \operatorname{loc\,Ind} M = 0.$$

Proof. Each point $(\alpha, p) \in M$ is contained in the open and closed set $M_\alpha \subset M$, and M is a subset of the zero-dimensional separable metrizable space $T_\alpha \times (I - Q_\alpha)$. Thus

$$\dim M_\alpha = \operatorname{ind} M_\alpha = \operatorname{Ind} M_\alpha = 0,$$

and hence $$\operatorname{ind} M = \operatorname{loc} \dim M = \operatorname{loc} \operatorname{Ind} M = 0,$$

as was to be shown.

[6.5] *For the space* M, $\dim M = 1$.

Proof. Since each finite covering of M has a special refinement and since a special covering has order not exceeding 1, therefore $\dim M \leqslant 1$.

Let G_0 be the set of points (α, p) of M with $p < 1$, and let G_1 be the set of points with $p > 0$. Then $\{G_0, G_1\}$ is a covering of M. Let $\{U_1,\ldots, U_r\}$ be any refinement of $\{G_0, G_1\}$.

Choose a special refinement of $\{U_i\}$. The set $(T'_\alpha \times W_1) \cap M$ is contained in some set U_i, and $U_i \subset G_0$. The set $(T'_\alpha \times W_k) \cap M$ is not contained in G_0 and hence is not contained in U_i. Hence there is a first j such that $(T'_\alpha \times W_j) \cap M \not\subset U_i$; let $(T'_\alpha \times W_j) \cap M \subset U_h$. Then, for any $p \in W_{j-1} \cap W_j$ and any β so large that $\beta > \alpha$ and $(\beta, p) \in M$, we have

$$(\beta, p) \in (T'_\alpha \times W_j) \cap M \subset U_h, \qquad (\beta, p) \in (T'_\alpha \times W_{j-1}) \cap M \subset U_i.$$

Thus $(\beta, p) \in U_i \cap U_h$, and the order of $\{U_i\}$ is at least one. Therefore $\dim M \geqslant 1$, and hence $\dim M = 1$.

[6.6] $\operatorname{Ind} M = 1$.

Proof. Let $F \subset U$ with F closed in M and U open in M. Choose a special refinement of the covering $\{M - F, U\}$ of M. Let V be the union of the sets of the special refinement which meet F; then $F \subset V \subset U$.

For each $j = 1,\ldots, k$, $\overline{W}_j - W_j$ consists of at most two points. Let $E = \bigcup_i (\overline{W}_j - W_j)$; then E is a finite subset of I. It is known that $\operatorname{Ind} T = 0$, and it follows that $\operatorname{Ind}(T \times E) = 0$. But $(\overline{V} - V) \cap M$ is a closed subset of $T \times E$; hence $\operatorname{Ind}((\overline{V} - V) \cap M) \leqslant 0$. Hence $\operatorname{Ind} M \leqslant 1$. Therefore $1 = \dim M \leqslant \operatorname{Ind} M \leqslant 1$, and hence $\operatorname{Ind} M = 1$, as was to be shown.

7. More examples

Example N. Let N be the space M^* formed from M by adding a single point x_0 as in § 5 above. A basic set of neighbourhoods of x_0 in N consists of the sets $(x_0) \cup M'_\alpha$ for $\alpha \in T$, where $M'_\alpha = (T'_\alpha \times I) \cap M$.

[7.1] *The space N is a normal Hausdorff space such that*

$$\dim N = \operatorname{Ind} N = 0.$$

Proof. It follows from [5.1] that N is a normal Hausdorff space and that $\dim N \leqslant \operatorname{loc}\dim M$. Hence, by [6.4], $\dim N \leqslant 0$ and hence, since N is not empty, $\dim N = 0$. This implies [(**6**) appendix] that $\operatorname{Ind} N = 0$.

Example N shows that the subset theorem does not hold for all normal Hausdorff spaces, even if the subset is required to be normal. For N is a normal Hausdorff space with $\dim N = \operatorname{Ind} N = 0$, having as an open subspace a normal space M with $\dim M = \operatorname{Ind} M = 1$.

Example Q. Let Q be a space consisting of a sequence $\{N_i\}$ of different copies of the space N together with a special point y_0. A basis for the open sets of Q is formed by the open sets of each N_i together with the sets $(y_0) \cup \bigcup_{i>j} N_i$ for $j = 1, 2, \ldots$.

[7.2] *The space Q is a normal Hausdorff space and* $\dim Q = \operatorname{Ind} Q = 0$.

Proof. If p and q are two points of N_j, then, since N_j is a Hausdorff space, p and q have disjoint neighbourhoods in N_j. If $p \in N_i$ and $q \in N_j$, then N_i and N_j are open, and $N_i \cap N_j = 0$. If $p = y_0$ and $q \in N_j$, then p and q have the disjoint neighbourhoods $(y_0) \cup \bigcup_{i>j} N_i$ and N_j. Thus Q is a Hausdorff space.

If E and F are disjoint closed sets of Q, then one of them, say F, does not contain y_0. Then y_0 has a neighbourhood which does not meet F, and hence $F \subset N_i \cup \ldots \cup N_j$ for some finite j. Since N_i is normal, there exist disjoint open sets U_i and V_i of N_i with $E \cap N_j \subset U_i$ and $F \cap N_i \subset V_i$. Let

$$U = U_i \cup \ldots \cup U_j \cup (y_0) \cup \bigcup_{i>j} N_j, \qquad V = V_1 \cup \ldots \cup V_j.$$

Then U and V are open, $E \subset U$, $F \subset V$, and $U \cap V = 0$. Thus Q is a normal space.

If F is any closed set of Q which does not meet (y_0), then F is a closed set of $N_1 \cup \ldots \cup N_j$ for some j and hence

$$\dim F \leqslant \dim(N_1 \cup \ldots \cup N_j).$$

Hence, by [7.1] and [2.5], $\dim F \leqslant 0$. Therefore, by [2.1], since

$$\dim(y_0) = 0,$$

we have $\dim Q = 0$. It follows [(**6**) appendix] that $\operatorname{Ind} Q = 0$, which completes the proof.

Example P. Let P be a space consisting of a sequence $\{M_i\}$ of different copies of the space M together with a special point y_0. A basis for the open sets of P is formed by the open sets of each M_i together with the sets $(y_0) \cup \bigcup_{i>j} M_i$ for $j = 1, 2, \ldots$.

[7.3] *The space P is a normal Hausdorff space and* $\operatorname{ind} P = 0$ *while* $\operatorname{loc}\dim P = \operatorname{loc}\operatorname{Ind} P = \dim P = \operatorname{Ind} P = 1$.

Proof. That P is a normal Hausdorff space is shown as in the proof of [7.2]. Since $\operatorname{ind} M = 0$, each point of M_i has an arbitrarily small open and closed neighbourhood in M_i. The point y_0 has arbitrarily small open and closed neighbourhoods of the form $(y_0) \cup \bigcup_{i>j} M_i$. Thus $\operatorname{ind} P = 0$.

The point y_0 has a neighbourhood U such that $\dim \overline{U} \leqslant \operatorname{loc}\dim P$. And the neighbourhood U contains a neighbourhood of the form $(y_0) \cup \bigcup_{i>j} M_i$, and hence contains the closed set M_{j+1}. Therefore

$$\dim \overline{U} \geqslant \dim M_{j+1} = 1.$$

Thus $\operatorname{loc}\dim P \geqslant 1$.

If $F \subset U$ with F closed and U open in P, then y_0 has a neighbourhood $(y_0) \cup \bigcup_{i>j} M_i$ which either does not meet F or is contained in U. Since $\operatorname{Ind} M_i = 1$, there is an open set V_i with boundary $B_i = \overline{V}_i - V_i \subset M_i$ such that $F \cap M_i \subset V_i \subset U \cap M_i$ and $\operatorname{Ind} B_i \leqslant 0$. Let V be the union of the sets V_i for $i \leqslant j$ together with the open and closed set $(y_0) \cup \bigcup_{i>j} M_i$ in case the latter meets F. Then $F \subset V \subset U$ and the boundary of V is $B = B_1 \cup \ldots \cup B_j$. Thus B is the union of disjoint relatively open and closed sets B_i with each $\operatorname{Ind} B_i \leqslant 0$. Hence [(**5**) proposition 5.1] $\operatorname{Ind} B \leqslant 0$. Therefore $\operatorname{Ind} P \leqslant 1$. Hence

$$1 \leqslant \operatorname{loc}\dim P \leqslant \dim P \leqslant \operatorname{Ind} P \leqslant 1,$$

$$1 \leqslant \operatorname{loc}\dim P \leqslant \operatorname{loc}\operatorname{Ind} P \leqslant \operatorname{Ind} P \leqslant 1.$$

This completes the proof.

Clearly P is a subspace of Q. Thus the subset theorem does not hold for the local dimension of normal Hausdorff spaces. For Q is a normal Hausdorff space with $\operatorname{loc}\dim Q = \operatorname{loc}\operatorname{Ind} Q = 0$, and P is a normal subspace of Q with $\operatorname{loc}\dim P = \operatorname{loc}\operatorname{Ind} P = 1$.

Also, though, by [4.3], the finite-sum theorem holds for the local dimension of normal spaces, the countable-sum theorem does not hold. For the normal space P is the union of a sequence of closed sets

$$(y_0),\ M_1,\ M_2, \ldots$$

with $\operatorname{loc}\dim (y_0) = 0$, $\operatorname{loc}\dim M_i = 0$, but $\operatorname{loc}\dim P = 1$.

Example S. O. V. Lokucievskiĭ (**8**) has given an example of a normal Hausdorff compact space S which is the union of two closed subsets S_1 and S_2 such that $\operatorname{ind} S_1 = \operatorname{Ind} S_1 = 1$, $\operatorname{ind} S_2 = \operatorname{Ind} S_2 = 1$, and $\operatorname{ind} S = \operatorname{Ind} S = 2$. Hence, by [1.7], $\operatorname{loc}\operatorname{Ind} S_1 = \operatorname{loc}\operatorname{Ind} S_2 = 1$, and

loc ind $S = 2$. Thus not even the finite-sum theorem holds for ind, loc Ind, and Ind.

Since $\dim S_1 \leqslant \operatorname{Ind} S_1 = 1$ and $\dim S_2 \leqslant \operatorname{Ind} S_2 = 1$, therefore, by the sum theorem, $\dim S \leqslant 1$. Also, S contains a closed set homeomorphic to a line segment; hence $\operatorname{loc}\dim S \geqslant 1$. Therefore

$$\operatorname{loc}\dim S = \dim S = 1.$$

[7.4] *No relations between* ind, loc dim, loc Ind, dim, *and* Ind, *other than those listed in* [1.7] *above, hold for all normal regular spaces.*

Proof. This is shown by the properties of Examples M, P, and S above, as is more clearly seen in the following table:

Space	ind	loc dim	loc Ind	dim	Ind
M	0	0	0	1	1
P	0	1	1	1	1
S	2	1	2	1	2

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ASSOCIATED STURM–LIOUVILLE SYSTEMS

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1. LET the regular Sturm–Liouville system

$$\begin{cases} y''+\{\lambda-q(x)\}y = 0 \quad (0 < x < 1), & \text{(A)} \\ y'(0) = h^{(0)}y(0), \qquad y'(1) = h^{(1)}y(1) & \text{(B)} \end{cases}$$

have eigenvalues $\lambda_0 < \lambda_1 < \lambda_2$, etc., and eigenfunctions ϕ_s corresponding to λ_s. Let $q(x)$ be repeatedly differentiable in $(0, 1)$; then the ϕ_s also are repeatedly differentiable; let W_{ns} be the Wronskian of the $n+1$ functions $\phi_0, \phi_1,..., \phi_{n-1}, \phi_s$ and let W_n be the Wronskian of the n functions $\phi_0, \phi_1,..., \phi_{n-1}$. Then, if $n \geqslant 1$ and

$$\phi_{ns} = W_{ns}/W_n,$$

the functions ϕ_{ns} $(s \geqslant n)$ are the eigenfunctions, with eigenvalues λ_s, of the system

$$\begin{cases} y''+\{\lambda-q_n(x)\}y = 0 \quad (0 < x < 1), & (\mathrm{A}_n) \\ \lim_{x\to 0} y(x) = 0, \qquad \lim_{x\to 1} y(x) = 0, & (\mathrm{B}_n) \end{cases}$$

where
$$q_n(x) = q(x)-2\frac{d^2}{dx^2}\log W_n. \qquad (\mathrm{C}_n)$$

For $n = 1$, the system $(\mathrm{A}_n, \mathrm{B}_n)$ is regular; but, for $n > 1$,

$$q_n(x) \sim \begin{cases} n(n-1)x^{-2} & (x \to 0), \\ n(n-1)(1-x)^{-2} & (x \to 1). \end{cases}$$

Inside $(0, 1)$, W_n is non-zero and q_n is continuous. For $s < n$, $\phi_{ns} \equiv 0$; for $s > n$, ϕ_{ns} has exactly $s-n$ zeros inside $(0, 1)$. The family ϕ_{ns} $(s \geqslant n)$ is L^2-closed and complete over $(0, 1)$.

The system $(\mathrm{A}_n, \mathrm{B}_n)$ may be called the 'nth system associated with the system (A, B)'. In this note the above statements are established, and examples are given of systems associated with non-regular Sturm–Liouville systems.

If $q(x)$ is continuous but not differentiable, the ϕ_s are differentiable twice only, and the Wronskians do not exist; however, when the Wronskians W_{ns}, W_n exist, they are equal to the modified Wronskians W^*_{ns}, W^*_n obtained by replacing $\phi_s^{(2k)}$ by $(-\lambda_s)^k\phi_s$, and $\phi_s^{(2k+1)}$ by $(-\lambda_s)^k\phi'_s$; the W^*_n are at least twice differentiable, and the statements above are

true for non-differentiable continuous q provided that the W are replaced by W^*.

2. The case $n = 1$

We have $W_1 = \phi_0$, of constant sign† for $0 \leqslant x \leqslant 1$; and

$$\phi_{1s} = \phi_s' - \frac{\phi_0'}{\phi_0}\phi_s = \phi_s' - v\phi_s, \quad \text{say,} \tag{D$_1$}$$

where
$$v' + v^2 = q - \lambda_0. \tag{E}$$

Then
$$\frac{d}{dx}(\phi_0 \phi_{1s}) = \phi_0 \phi_s'' - \phi_0'' \phi_s = (\lambda_0 - \lambda_s)\phi_0 \phi_s. \tag{F$_1$}$$

Since
$$\phi_{1s}(0) = 0 = \phi_{1s}(1), \tag{G}$$

we have
$$\phi_0 \phi_{1s} = (\lambda_0 - \lambda_s) \int\limits_0^x \phi_0(\xi)\phi_s(\xi)\, d\xi = -(\lambda_0 - \lambda_s) \int\limits_x^1 \phi_0 \phi_s\, d\xi. \tag{G$'$}$$

Hence
$$\begin{aligned} \phi_{1s}' &= (\lambda_0 - \lambda_s)\phi_s - v\phi_{1s}, \\ \phi_{1s}'' &= (\lambda_0 - \lambda_s)\phi_s' - v'\phi_{1s} - v\{(\lambda_0 - \lambda_s)\phi_s - v\phi_{1s}\} \\ &= (\lambda_0 - \lambda_s - v' + v^2)\phi_{1s} \\ &= (q_1 - \lambda_s)\phi_{1s}, \end{aligned}$$

where
$$q_1 = \lambda_0 - v' + v^2 = q - 2v' = q - 2\frac{d^2}{dx^2}(\log W_1).$$

Now from (D_1),
$$\phi_{1s}/\phi_0 = \frac{d}{dx}(\phi_s/\phi_0);$$

since ϕ_s has exactly s zeros† inside $(0, 1)$, by Rolle's theorem, ϕ_{1s} has at least $s-1$. But from (F_1) and (G) and Rolle's theorem, ϕ_{1s} has at most $s-1$ zeros inside $(0, 1)$; hence it has $s-1$ exactly. It follows† that the ϕ_{1s} $(s \geqslant 1)$ are all the eigenfunctions of the regular system $(\mathrm{A}_1, \mathrm{B}_1)$. For $\lambda \neq \lambda_0$ the general solution of (A_1) is

$$X_1 = W(\phi_0, \chi)/W_1,$$

where χ is the general solution of (A). For $\lambda = \lambda_0$, $W(\phi_0, \chi)$ is constant and one solution of (A_1) is $1/\phi_0$; two independent solutions are

$$\frac{1}{\phi_0} \int\limits_0^x \phi_0^2(\xi)\, d\xi, \qquad \frac{1}{\phi_0} \int\limits_x^1 \phi_0^2(\xi)\, d\xi.$$

It is easily verified that the only solutions of (A_1) which satisfy (G) are the ϕ_{1s} $(s \geqslant 1)$.

† E. L. Ince, *Ordinary Differential Equations* (London, 1927), § 10.61, 235.

3. The case $n > 1$

Applying Jacobi's theorem to the determinant W_{ns}, we have, for $n > 1$,

$$W_{ns}W_{n-1} = W_n \frac{d}{dx} W_{n-1,s} - W_{n-1,s} \frac{d}{dx} W_n,$$

with a similar relation with W^* for W. Hence

$$\phi_{ns} = \frac{W_{ns}}{W_n} = \frac{1}{W_{n-1}} \frac{d}{dx}(W_{n-1}\phi_{n-1,s}) - \phi_{n-1,s}\frac{1}{W_n}\frac{d}{dx}W_n$$

$$= \phi'_{n-1,s} - v_{n-1}\phi_{n-1,s} = \frac{1}{\phi_{n-1,n-1}} W(\phi_{n-1,n-1}, \phi_{n-1,s}), \qquad (\mathrm{D}_n)$$

where $\quad v_n = \phi'_{nn}/\phi_{nn}, \qquad v_{n-1} = W'_n/W_n - W'_{n-1}/W_{n-1}.$

Hence, by steps similar to those of § 2, and by induction on n,

$$v'_n + v_n^2 = q_n - \lambda_n, \qquad (\mathrm{E}_n)$$

$$\frac{d}{dx}(\phi_{n-1,n-1}\phi_{ns}) = (\lambda_{n-1} - \lambda_s)\phi_{n-1,n-1}\phi_{n-1,s}, \qquad (\mathrm{F}_n)$$

$$\phi''_{ns} = (q_n - \lambda_s)\phi_{ns}, \qquad q_n = q_{n-1} - 2v'_{n-1},$$

$$q_n + 2\frac{d}{dx}\left(\frac{W'_n}{W_n}\right) = q_{n-1} + 2\frac{d}{dx}\left(\frac{W'_{n-1}}{W_{n-1}}\right) = q.$$

We now prove by induction on n the following:

$$\phi_{ns} = C_{ns}\prod_{t=0}^{n-1}(\lambda_t - \lambda_s)x^n\{1 + O(x^2)\} \quad (C_{ns} \neq 0), \qquad (\mathrm{G}_n)$$

$$\phi'_{ns} = nx^{-1}\phi_{ns}\{1 + O(x^2)\}, \qquad (\mathrm{H}_n)$$

$$v_n = nx^{-1}\{1 + O(x^2)\}, \qquad (\mathrm{J}_n)$$

all as $x \to 0$, with similar relations as $x \to 1$;

$$\phi_{ns} \text{ has } s-n \text{ zeros inside } (0, 1). \qquad (\mathrm{K}_n)$$

By (K_n), ϕ_{nn}, and so also W_{n+1}, is non-zero inside $(0, 1)$, so that q_{n+1} and $\phi_{n+1,s}$ are continuous inside $(0, 1)$. First, by (G) and (G′), as $x \to 0$,

$$\phi_{1s}(x) \sim (\lambda_0 - \lambda_s)\phi_s(0)x;$$

also $\qquad \phi''_{1s}(0) = (q_1 - \lambda_s)\phi_{1s}(0) = 0,$

which together imply (G_1); (H_1) follows from (G_1) and (F_1), together with

$$\phi_s = \phi_s(0)\{1 + h^{(0)}x + O(x^2)\};$$

and (J_1) is a case of (H_1). It remains to deduce (G_{n+1}) to (K_{n+1}) from (G_n) to (K_n). First, by (D_{n+1}), (H_n), (J_n),

$$\phi_{n+1,s} = \phi_{ns}\left[\frac{n}{x} + O(x) - \frac{n}{x} + O(x)\right] = o(1) \quad (x \to 0).$$

Hence $$\phi_{nn}\phi_{n+1,s} = (\lambda_n-\lambda_s)\int\limits_0^x \phi_{nn}\phi_{ns}\,d\xi,$$
whence we have (G_{n+1}) with
$$C_{n+1,s} = C_{ns}/(2n+1) \neq 0.$$
By differentiating this last we obtain (H_{n+1}), of which (J_{n+1}) is a special case.

From (D_{n+1}) and (K_n), $\phi_{n+1,s}$ has at least $s-n-1$ zeros inside $(0,1)$; from (F_{n+1}), (K_n), (G_n), it has at most $s-n-1$ zeros inside $(0,1)$; hence (K_{n+1}) is deduced.

Lastly we may prove that, as $x \to 0$,
$$q_n(x) = n(n-1)x^{-2}+O(1), \tag{L_n}$$
with a similar relation as $x \to 1$. For, given (L_n) and (J_n),
$$q_{n+1} = q_n-2v_n' = 2\lambda_n+2v_n^2-q_n = O(1)+n(n+1)x^{-2},$$
which is (L_{n+1}).

For $\lambda \neq \lambda_s$ $(s < n)$ the general solution of (A_n) is
$$y = \chi_n = W(\phi_0,\phi_1,\ldots,\phi_{n-1},\chi)/W_n,$$
where χ is the general solution of (A). For $\lambda = \lambda_{n-1}$ a solution is
$$y = \frac{1}{\phi_{n-1,n-1}}W(\phi_{n-1,n-1},\chi_{n-1,n-1}) = \frac{C}{\phi_{n-1,n-1}} = C\frac{W(\phi_0,\phi_1,\ldots,\phi_{n-2})}{W(\phi_0,\phi_1,\ldots,\phi_{n-1})}.$$
For $\lambda = \lambda_s$, $s \leqslant n-1$, a solution is
$$y = \psi_{ns} = W_n^{(s)}/W_n,$$
where $W_n^{(s)}$ is the Wronskian of the $n-1$ functions
$$\phi_t \quad (0 \leqslant t \leqslant n-1;\ t \neq s).$$

4. Since the system (A_n, B_n) is not regular for $n > 1$, it remains to prove that the family ϕ_{ns} $(s \geqslant n)$ is L^2-complete over $(0,1)$; this implies incidentally that the ϕ_{ns} are the only bounded solutions of (A_n). Since (A_1, B_1) is regular, it is sufficient to verify that the completeness of the family ϕ_{ns} implies that of the family $\phi_{n+1,s}$.

Let $f(x)$ be of $L^2(0,1)$; then, given $\epsilon > 0$, there exists $g(x)$ such that

(i) $g(x) = 0 \quad (0 < x < \delta;\ 1-\delta < x < 1;\ \delta > 0)$,

(ii) $g'(x)$ is continuous in $(0,1)$,

(iii) $\int\limits_0^1 |f-g|^2\,d\xi < \epsilon$.

Then, if $$h = g'+v_n g, \qquad \phi_{nn}h = \frac{d}{dx}(\phi_{nn}g),$$

h is of $L^2(0,1)$; also

$$\int_0^1 h\phi_{nn}\,d\xi = [g\phi_{nn}]_0^1 = 0,$$

so that, assuming the completeness of the family ϕ_{ns}, we have

$$h = \sum_{s=n+1}^{N} c_s\phi_{ns}+\eta,$$

where
$$\int_0^1 |\eta^2|\,dx < \epsilon.$$

Now

$$\phi_{nn}g = \int_0^x \phi_{nn}h\,d\xi = \sum_{s=n+1}^{N} c_s\int_0^x \phi_{nn}\phi_{ns}\,d\xi + \int_0^x \phi_{nn}\,\eta\,d\xi$$

$$= \phi_{nn}\sum_{s=n+1}^{N} C_s\phi_{n+1,s}+\phi_{nn}\,\zeta,$$

where $C_s = c_s(\lambda_n-\lambda_s)^{-1}$, and

$$\zeta = \frac{1}{\phi_{nn}}\int_0^x \phi_{nn}\,\eta\,d\xi = -\frac{1}{\phi_{nn}}\int_x^1 \phi_{nn}\,\eta\,d\xi;$$

since, by (G_n) and its analogue for $x\to 1$,

$$\int_0^x \phi_{nn}^2\,dx = O(\phi_{nn}^2), \qquad \int_x^1 \phi_{nn}^2\,dx = O(\phi_{nn}^2)$$

when $x\to 0$, 1, respectively, we have by Schwarz's inequality

$$|\zeta^2| < M_n\int_0^1 |\eta^2|\,dx < M_n\,\epsilon, \qquad \int_0^1 |\zeta^2|\,dx < M_n\,\epsilon.$$

Hence the result.

5. Examples

(1) If $q(x) = 0$, $h^{(0)} = 0 = h^{(1)}$, then $\lambda_s = (2\pi s)^2$, $\phi_s = \cos 2\pi sx$ $(s = 0, 1, 2,...)$. Since $v = 0$, $q_1 = q$ and

$$\phi_{1s} = \phi_s' = 2\pi s\sin 2\pi sx \quad (s = 1, 2,...).$$

For $n > 1$, ϕ_{ns} is obtainable as in Example 3.

(2) If $q(x) = x^2$ and the interval is $(-\infty,\infty)$, (A) is $y''+(\lambda-x^2)y = 0$, with $\phi_0 = e^{-\frac{1}{2}x^2}$, $\lambda_0 = 1$. Since $v = x$, $q_1 = x^2-2$; hence†

$$\lambda_{s+1} = \lambda_s+2, \qquad \phi_{1s} = k_s\phi_{s-1}.$$

† Compare P. A. M. Dirac, *Quantum Mechanics* (3rd ed., Oxford, 1947), § 34, 136–9.

The associated systems are all identical, $\lambda_s = 2s+1$, and, since

$$\phi_0 \phi_s = \frac{1}{\lambda_0-\lambda_s}\frac{d}{dx}(\phi_0 \phi_{1s}) = \frac{k_s}{2s}\frac{d}{dx}(\phi_0 \phi_{s-1}),$$

it follows that

$$\phi_s = K_s \phi_0^{-1}\left(\frac{d}{dx}\right)^s \phi_0^2 = K_s e^{\frac{1}{2}x^2}\left(\frac{d}{dx}\right)^s e^{-x^2}.$$

(3) The Legendre functions†

$$y_s = (\sin\theta)^{\frac{1}{2}} P_s(\cos\theta) \quad (0 < \theta < \pi)$$

satisfy
$$y''+(\lambda+\tfrac{1}{4}\operatorname{cosec}^2\theta)y = 0,$$

where
$$\lambda_s = (s+\tfrac{1}{2})^2 \quad (s = 0, 1, 2,\ldots).$$

Writing $\mu = \cos\theta$, and $W_{(\mu)}$ for the Wronskians with respect to μ, we get

$$\begin{aligned} W_n = W(y_0, y_1,\ldots, y_{n-1}) &= \left(\frac{d\mu}{d\theta}\right)^{\frac{1}{2}n(n-1)} W_{(\mu)}(y_0, y_1,\ldots, y_{n-1}) \\ &= \left(\frac{d\mu}{d\theta}\right)^{\frac{1}{2}n(n-1)} (\sin\theta)^{\frac{1}{2}n} W_{(\mu)}(P_0, P_1, \ldots, P_{n-1}) \\ &= A_n(\sin\theta)^{\frac{1}{2}n^2}, \end{aligned}$$

and similarly
$$W_{ns} = A_n(\sin\theta)^{\frac{1}{2}(n+1)^2}\left(\frac{d}{d\mu}\right)^n P_s(\mu).$$

Hence‡
$$\phi_{ns} = (\sin\theta)^{n+\frac{1}{2}}\left(\frac{d}{d\mu}\right)^n P_s(\mu) = (\sin\theta)^{\frac{1}{2}} P_s^{(n)}(\mu).$$

(4) For the Hankel system§ of order ν,

$$y = \phi_k(x) = c_k(kx)^{\frac{1}{2}} J_\nu(kx), \qquad \phi_0(x) = x^{\nu+\frac{1}{2}},$$

$$y''+\left(\lambda-\frac{\nu^2-\frac{1}{4}}{x^2}\right)y = 0, \qquad \lambda = k^2.$$

Here $v = \phi_0'/\phi_0 = (\nu+\frac{1}{2})/x$, whence

$$q_1 = \frac{(\nu+1)^2-\frac{1}{4}}{x^2}$$

and the first associated system is the Hankel system of order $\nu+1$.

6. As a corollary of the main theorem, if

$$S(x) = \sum_0^n c_s \phi_s(x),$$

† E. C. Titchmarsh, *Eigenfunction Expansions* (Oxford, 1946), § 4.5, 64.

‡ E. T. Whittaker and G. N. Watson, *Modern Analysis* (3rd ed., Cambridge, 1927), § 15.5, 323.

§ Titchmarsh, op. cit. § 4.8, 70, and § 4.11, 75.

then $S(x)$ has at most n zeros in $(0, 1)$. This result is due to Kellogg.† For, if $S(x)$ has k zeros, then by Rolle's theorem

$$S_1(x) = \phi_0 \frac{d}{dx}\left\{\phi_0^{-1} \sum_0^n c_s \phi_s(x)\right\} = \sum_1^n c_s \phi_{1s}$$

has at least $k-1$ zeros inside $(0, 1)$; by induction

$$S_m(x) = \sum_m^n c_s \phi_{ms}$$

has at least $k-m$ zeros, and $S_n(x) = c_n \phi_{nn}$ has at least $k-n$; since ϕ_{nn} is non-zero, either $k \leqslant n$ or $c_n = 0$; but, if $c_n = 0$, then $k \leqslant n-1 < n$.

This proof of the corollary depends only on the fact that the Wronskians W_n are non-zero. If $\phi_s = e^{\alpha_s x}$, where the α_s are any distinct real numbers, then the W_n are all non-zero, and so $S(x)$ has at most n real zeros.

7. If (A, B) is given, the associated systems $(\mathrm{A}_n, \mathrm{B}_n)$ are uniquely defined; but to a given $(\mathrm{A}_n, \mathrm{B}_n)$ belong an infinity of (A, B). For example, given $(\mathrm{A}_1, \mathrm{B}_1)$ we may solve for v

$$\lambda_0 - v' + v^2 = q_1,$$

with any λ_0 such that $\lambda_0 < \lambda_1$; then, if

$$\phi_0 = \exp\left(\int_0^x v\, d\xi\right), \qquad (\lambda_0 - \lambda_s)\phi_0 \phi_s = \frac{d}{dx}(\phi_0 \phi_{1s}),$$

it will follow that the ϕ_s are the eigenfunctions of (A, B) with

$$q = q_1 + 2v', \qquad h^{(0)} = v(0), \qquad h^{(1)} = v(1).$$

For example, if

$$q_1 = 0, \qquad \lambda_s = (2\pi s)^2, \qquad \phi_{1s} = \sin 2\pi s x,$$

we can take

$$\lambda_0 = -\rho^2, \qquad \phi_0 = \operatorname{sech} \rho(x-\alpha), \qquad v = -\rho \tanh \rho(x-\alpha),$$

$$q(x) = -2\rho^2 \operatorname{sech}^2 \rho(x-\alpha),$$

$$\phi_s(x) = 2\pi s \cos 2\pi s x - \rho \tanh \rho(x-\alpha) \sin 2\pi s x.$$

Starting from a given $(\mathrm{A}_n, \mathrm{B}_n)$ we can similarly construct an (A, B) with arbitrary $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ (provided only that $\lambda_{s+1} > \lambda_s$). Thus there exists a regular Sturm–Liouville system with any finite set of real numbers as eigenvalues.

† O. D. Kellogg, *American Journal of Mathematics*, (i) 'Oscillations of functions of an orthogonal set' (1916) 1, (ii) 'Orthogonal sets arising from integral equations' (1918) 145, (iii) 'Interpolation properties of orthogonal sets of solutions of differential equations' (1918) 225. Kellogg uses the functional determinants $\det\{\phi_s(x_t)\}$, not the Wronskians W_{ns} or W_{ns}^*.

SOME REMARKS ON TAUBERIAN CONDITIONS

By R. C. BUCK (*University of Wisconsin*)

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THE following is a familiar property of positive series.

THEOREM A. *If $\sum a_n$ converges and $\{a_n\} \downarrow 0$, then $\lim_{n\to\infty} na_n = 0$.*

In this note, we examine the role played by the condition '$\{a_n\} \downarrow 0$'. Szász (**5**) showed that Theorem A holds if this condition is weakened to the requirement that $\{a_n\}$ be *quasi-monotonic*, i.e. that $\{a_n\}$ obey

$$0 \leqslant a_{n+1} \leqslant \left(1+\frac{\alpha}{n}\right)a_n \tag{1}$$

for some constant $\alpha > 0$ and all $n = 1, 2,\ldots$. Clunie (**2**) showed recently that the condition (1) is best-possible in the sense that, if α is replaced by an increasing function of n, $\sum a_n$ may converge without $\lim_{n\to\infty} na_n = 0$. However, condition (1) can be generalized in a different way to preserve Theorem A. Let us first observe that, if $\{a_n\}$ obeys (1), then, for any $k > n$,

$$a_k \leqslant \left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha}{n+1}\right)\ldots\left(1+\frac{\alpha}{k-1}\right)a_n \leqslant \alpha\left\{\log\frac{k}{n}\right\}a_n.$$

In particular, whenever $\{a_n\}$ is quasi-monotonic and $\lambda > 1$, there is a number M such that

$$0 \leqslant a_k \leqslant Ma_n \tag{2}$$

whenever $n \leqslant k \leqslant \lambda n$.

DEFINITION. *A sequence $\{a_n\}$ is of 'bounded increase' if it obeys condition* (2) *for some choice of M and $\lambda > 1$.*

Any quasi-monotonic sequence has bounded increase, but not conversely; if $0 < b \leqslant a_n \leqslant B$ for all n, then $\{a_n\}$ has bounded increase. So also for the sequences $a_n = n^p$ for any real number p. If $\{a_n\}$ and $\{b_n\}$ have bounded increase, so does $\{a_n b_n\}$.

The generalization of Theorem A is

THEOREM B. *If $\sum a_n$ converges and $\{a_n\}$ has bounded increase, then $\lim_{n\to\infty} na_n = 0$.*

Rather than give the simple and direct proof of this, I prefer to approach it from a different direction which will cast more light on the nature of the restriction imposed on $\{a_n\}$. We first observe that, if $\sum a_n$ converges, then (with no restriction on $\{a_n\}$)

$$(C,1)\text{-}\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{1}{n}\sum_1^n ka_k = 0.$$

This suggests that we are actually dealing with a Tauberian situation relating to Cesàro summability. Put $S_n = na_n$. Then, if $\{S_n\}$ were to obey the proper Tauberian condition, we might conclude that $\lim S_n = 0$, which is the conclusion of Theorems A and B.

THEOREM C. *If* $(C,1)\text{-}\lim S_n = 0$ *and* $\{S_n\}$ *is of bounded increase, then* $\lim_{n\to\infty} S_n = 0$.

The condition on $\{S_n\}$ is not a standard one, and the theorem itself is deceptive, for 0 cannot be replaced by any other value. For example, $S_n = 2+(-1)^n$ is Cesàro-summable to 2 and is of bounded increase, but does not converge. The reason for the special nature of the limit is to be found in the following result, whose proof appears elsewhere (**1**): *if* $\liminf_{n\to\infty} S_n \geqslant 0$ *and* $(C,1)\text{-}\lim S_n = 0$, *then* $\{S_n\}$ *converges to* 0 *on a set of indices of unit density.*

Using this result, we reduce our considerations to the following problem. Suppose that a sequence $\{S_n\}$ is known to have a sub-sequence $\{S_{r_n}\}$ which converges to a limit L. What additional conditions on the sequence $\{S_n\}$ and on the mode of distribution of the terms of the sub-sequence will ensure that $\lim S_n = L$? We say that L is a *strong limit point* for $\{S_n\}$ if there is a sub-sequence $\{S_{r_n}\}$ convergent to L for which $r_{n+1}/r_n = O(1)$.

THEOREM D. *If* 0 *is a strong limit point for* $\{S_n\}$ *and* $\{S_n\}$ *has bounded increase, then* $\lim S_n = 0$.

Proof. Assume that $r_{n+1}/r_n \leqslant A$ for all n. Since $\{S_n\}$ has bounded increase, we may choose M and $\lambda > A$ so that $0 \leqslant S_k \leqslant MS_n$ whenever $n \leqslant k \leqslant \lambda n$. Given ϵ, choose N so that $S_{r_n} < \epsilon$ whenever $n \geqslant N$. Take any $k > r_N$ and choose $n > N$ so that

$$r_n \leqslant k \leqslant r_{n+1} \leqslant Ar_n < \lambda r_n.$$

Then, $$0 \leqslant S_k \leqslant MS_{r_n} < M\epsilon,$$

proving that $\lim_{k\to\infty} S_k = 0$.

At this point, we have completed the proof of Theorems B and C.

Other conditions may also be used in Theorem D. Let us say that L is a *very strong limit point* for $\{S_n\}$ if $\lim_{n=\infty} S_{r_n} = L$ where $\lim_{n\to\infty} r_{n+1}/r_n = 1$. This includes the case of a sub-sequence of positive density, as well as more sparsely distributed sub-sequences such as $r_n = n^2$ or $r_n = n^{\log n}$.

THEOREM E. *If $\{S_n\}$ has L as a very strong limit point and $\{S_n\}$ is slowly oscillating, then* $\lim S_n = L$.

The property of being 'slowly oscillating' [(**3**) 125] means that for any $\epsilon > 0$ there exist numbers N and $\lambda > 1$ such that $|S_k - S_n| < \epsilon$ for all $n \geqslant N$ and all k with $n \leqslant k \leqslant \lambda n$. Choose N so that $|S_{r_n} - L| < \epsilon$ and $r_{n+1}/r_n < \lambda$ whenever $n \geqslant N$. As before, if $k > \lambda r_N$ and

$$r_n \leqslant k \leqslant r_{n+1} < \lambda r_n,$$

then
$$|S_k - L| \leqslant |S_k - S_{r_n}| + |S_{r_n} - L| < 2\epsilon,$$

proving that $\lim_{k\to\infty} S_k = L$.

A one-sided condition may also be used.

THEOREM F. *If $\{S_n\}$ has L as a very strong limit point and obeys the condition $(n+1)S_{n+1} - nS_n \geqslant -B$ for all n, then* $\lim_{n\to\infty} S_n = L$.

If $n \leqslant k \leqslant m$, then

$$mS_m + (m-k)B \geqslant kS_k \geqslant nS_n - (k-n)B,$$

so that, choosing $n = r_j$ and $m = r_{j+1}$, we have

$$\frac{r_{j+1}}{r_j} S_{r_{j+1}} + \left(\frac{r_{j+1}}{r_j} - 1\right) B \geqslant S_k \geqslant \frac{r_j}{r_{j+1}} S_{r_j} - \left(1 - \frac{r_j}{r_{j+1}}\right) B.$$

Since $\lim_{j\to\infty} S_{r_j} = L$ and $\lim_{j\to\infty} \frac{r_{j+1}}{r_j} = 1$, the conclusion $\lim_{k\to\infty} S_k = L$ follows at once.

Such 'density' Tauberian theorems have application outside the immediate area of series.

THEOREM G. *Let f be an entire function of exponential type $c < \pi$ which is bounded on the real axis, or, more generally, which obeys the condition*

$$\int_1^\infty x^{-2} \log|f(x)f(-x)|\, dx < \infty. \tag{3}$$

Suppose that f has no zero among the positive integers, and obeys the inequality $|f(n+1)| \geqslant \delta |f(n)|$ for $n = 1, 2,\ldots$, where δ is a fixed positive number. Then,

$$\lim_{n\to\infty} n^{-1} \log|f(n)| = 0.$$

By the theorem due to Levinson (**4**), a function obeying (3) must satisfy $|f(n)| \geqslant e^{-\epsilon n}$ on a set of integers of positive density, for each $\epsilon > 0$. The sequence $\{S_n\}$, where $S_n = n^{-1}\log|f(n)|$, has then a subsequence of positive density which converges to zero. Since

$$|f(n+1)| \geqslant \delta|f(n)|,$$

we see that $$(n+1)S_{n+1}-nS_n \geqslant \log\delta = -B.$$

Applying the previous theorem, we have $\lim_{n\to\infty} S_n = 0$.

I conclude with an observation made by the referee, which shows again that 'bounded increase' is a natural extension of 'monotonic', and 'quasi-monotonic'.

THEOREM H. *Let $\{a_n\}$ have bounded increase, and let $\{r_n\}$ be an increasing sequence of integers such that*

$$\frac{r_{n+1}-r_n}{r_n-r_{n-1}} = O(1).$$

Then, $\sum a_n$ and $\sum (r_n-r_{n-1})a_{r_n}$ are either both convergent, or both divergent.

I omit the proof, which follows the usual lines of the standard condensation test.

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A HYPERGEOMETRIC IDENTITY

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1. In a brief note† printed a few years ago I gave the formula [loc. cit. (4)]

$$F(a,b;c;x) = h\sum_{n=0}^{\infty}\frac{(e)_n(h+\alpha n-n+1)_{n-1}}{n!\,(c)_n}\,{}_4F_3\begin{bmatrix}-n, & a, & b, & 1+h/\alpha\\ e, & h+\alpha n-n+1, & h/\alpha & \end{bmatrix}(-x)^n\times$$
$$\times\,{}_2F_1(e+n,h+\alpha n;c+n;x), \quad (1)$$

where I now replace my original α by $1-\alpha$. I want to give here first a simplification and then an extension of this formula (1).

The simplification is that, *when $eh = ab$, we can reduce the order of the ${}_4F_3$ by the formula*

$$(e+1)(h+\alpha n-n+1)\,{}_4F_3\begin{bmatrix}-n, & a, & b, & 1+h/\alpha\\ e, & h+n\alpha-n+1, & h/\alpha & \end{bmatrix}$$
$$= -(n-1)k\,{}_3F_2\begin{bmatrix}2-n, & a+1, & b+1\\ e+2, & h+n\alpha-n+2 & \end{bmatrix}, \quad (2)$$

where $ek = (e-a)(e-b)$.

To prove this, notice that, for any r (which will in fact be an integer, positive or zero), the condition on h and the definition of k together give

$$(a+r)(b+r)+kr = (e+r)(h+r),$$

while, evidently,

$$r(h+\alpha n-n+r)-n(h+r\alpha) = (r-n)(h+r).$$

Multiplying these two identities by $r-n$, $e+r$ respectively, subtracting, and rearranging, we get

$$r(e+r)(h+\alpha n-n+r)-(r-n)(a+r)(b+r)$$
$$= k(r-n)r+n(e+r)(h+r\alpha). \quad (3)$$

Now multiply the first member of this equation by

$$\frac{(1-n)_{r-1}(a+1)_{r-1}(b+1)_{r-1}}{r!\,(e+2)_{r-1}(h+\alpha n-n+2)_{r-1}} \quad (4)$$

and sum from $r = 0$ to $r = n$. We get

$$\sum_{r=0}^{n}\frac{(1-n)_{r-1}(a+1)_{r-1}(b+1)_{r-1}}{(r-1)!\,(e+2)_{r-2}(h+\alpha n-n+2)_{r-2}}-\sum_{r=0}^{n}\frac{(1-n)_r(a+1)_r(b+1)_r}{r!\,(e+2)_{r-1}(h+\alpha n-n+2)_{r-1}},$$

† T. W. Chaundy, 'Some hypergeometric identities', *J. of London Math. Soc.* 26 (1951), 42–4.

Quart. J. Math. Oxford (2), 6 (1955), 132–4.

which vanishes since there is zig-zag cancellation and the end-terms are zero. Thus, multiplying the second member of (3) by (4) and summing, we have

$$k(1-n)\sum_{r=0}^{n}\frac{(2-n)_{r-1}(a+1)_{r-1}(b+1)_{r-1}}{(r-1)!\,(e+2)_{r-1}(h+\alpha n-n+2)_{r-1}}$$

$$=\sum_{r=0}^{n}\frac{(-n)_r(a+1)_{r-1}(b+1)_{r-1}(h+r\alpha)}{r!\,(e+2)_{r-2}(h+\alpha n-n+2)_{r-1}}$$

$$=\frac{(e+1)(h+\alpha n-n+1)}{h}\sum_{r=0}^{n}\frac{(-n)_r(a)_r(b)_r(h+r\alpha)}{r!\,(e)_r(h+\alpha n-n+1)_r}$$

since $ab = eh$. This is the required identity.

2. The extension of (1) is to hypergeometric functions in two variables. The elementary F on the left is replaced by an Appell's function $F^{(1)}(a;b,b';c;x,y)$; that on the right by

$$F^{(1)}(e+m+n,h+\alpha m,h'+\beta n;c+m+n;x,y).$$

The ${}_4F_3$ on the right I need to extend into the analogous function in two variables (which are each given the value unity). For this I use the notation

$$F\left[\begin{matrix}a;\,b_1,b_2,b_3;\,b_1',b_2',b_3';\\ c\quad\ c_1,c_2\qquad c_1',c_2'\end{matrix}\,x,y\right]$$

$$\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_m(b_3)_m(b_1')_n(b_2')_n(b_3')_n}{(c)_{m+n}\,m!\,(c_1)_m(c_2)_m\,n!(c_1')_n(c_2')_n}x^my^n, \tag{5}$$

in which the semicolons separate the parameters with respective suffixes $m+n$, m, n (and I have displaced $m!$, $n!$ to preserve the pattern). In this notation I should write, for instance,

$$F\left[\begin{matrix}a;b,b';\\ c\end{matrix}\,x,y\right]$$

for Appell's $F^{(1)}(a;b,b';c;x,y)$. The extension of (1) is, then,

$$F\left[\begin{matrix}a;b;b';\\ c\end{matrix}\,x,y\right]$$

$$=hh'\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{(e)_{m+n}(h+\alpha m-m+1)_{m-1}(h'+\beta n-n+1)_{n-1}}{m!\,n!\,(c)_{m+n}}\times\right.$$

$$\times F\left[\begin{matrix}a; & -m, & b, & 1+h/\alpha; & -n, & b', & 1+h'/\beta\\ e & h+\alpha m-m+1, & h/\alpha & & h'+\beta n-n+1, & h'/\beta & \end{matrix}\right]\times$$

$$\left.\times(-x)^m(-y)^nF\left[\begin{matrix}e+m+n;h+\alpha m;h'+\beta n;\\ c+m+n\end{matrix}\,x,y\right]\right\}. \tag{6}$$

The proof is on all fours with that given originally for (1). Picking out the coefficient of

$$(a)_{M+N}(b)_M(b')_N/M!\,N!$$

on the two sides of (6) we need to prove that

$$x^M y^N/(c)_{M+N} = (h+\alpha M)(h'+\beta N)\times$$
$$\times \sum_{m=M}^{\infty}\sum_{n=N}^{\infty}\left\{\frac{(e)_{m+n}(h+\alpha m-m+1)_{m-1}(h'+\beta n-n+1)_{n-1}}{m!\,n!\,(c)_{m+n}}\times\right.$$
$$\times\frac{(-m)_M(-n)_N(-x)^m(-y)^n}{(e)_{M+N}(h+\alpha m-m+1)_M(h'+\beta n-n+1)_N}\times$$
$$\left.\times F\left[\begin{matrix}e+m+n;h+\alpha m;h'+\beta n;\\ c+m+n\end{matrix}x,y\right]\right\}.$$

Write $m = M+r$, $n = N+s$. Then we want

$$1 = (h+\alpha M)(h'+\beta N)\times$$
$$\times\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left\{\frac{(e+M+N)_{r+s}\{h+\alpha(M+r)-r+1\}_{r-1}\{h'+\beta(N+s)-s+1\}_{s-1}}{r!\,s!\,(c+M+N)_{r+s}}\times\right.$$
$$\left.\times(-x)^r(-y)^s F\left[\begin{matrix}e+M+N+r+s;h+\alpha(M+r);h'+\beta(N+s);\\ c+M+N+r+s\end{matrix}x,y\right]\right\}.$$

When we take the terms independent of y or independent of x in this, we get the corresponding formula in one variable already proved in the original note. We thus need only to pick out on the right the coefficient of any $x^R y^S$ (R, $S>0$) and to prove for it that

$$0=\sum_{r=0}^{R}\sum_{s=0}^{S}\frac{\{h+\alpha(M+r)-r+1\}_{R-1}\{h'+\beta(N+s)-s+1\}_{S-1}}{r!\,s!\,(R-r)!\,(S-s)!}.$$

But the double sum on the right is the coefficient of $x^{R-1}y^{S-1}$ in the expansion in ascending powers of x, y of

$$(RS)^{-1}(1-x)^{-(h+\alpha M+1)}(1-y)^{-(h'+\beta N+1)}\{1-(1-x)^{1-\alpha}\}^R\{1-(1-y)^{1-\beta}\}^S,$$

and this is evidently zero since the lowest power on the right is $x^R y^S$.

It is fairly obvious that by an analogous proof we could extend (6), *mutatis mutandis*, to hypergeometric functions in any number of variables.

I do not see in these extended cases any possibility of simplification in the hypergeometric factor on the right similar to that given in § 1.

ON LIMITS OF THE AREA OF A POLYGON INSCRIBED IN A SIMPLE CLOSED CURVE

By A. S. BESICOVITCH (*Cambridge*)

[Received 3 January 1955]

GIVEN a simple closed curve Γ we shall call a finite sequence $C_1, C_2,..., C_n$ of points on it *monotone* if the points are met in order of increasing or decreasing suffixes when we describe Γ in a constant direction. The polygon $C_1 C_2 ... C_n C_1$ will be called *inscribed* in Γ if the points $C_1, C_2,..., C_n$ form a monotone sequence on Γ. If Γ is convex, then any inscribed polygon is simple, i.e. has no double points, but in the general case this is not true. The points $C_1, C_2,..., C_n$ divide Γ into n arcs subtended by the sides of the polygon. When we speak about an inscribed polygon with *small* sides, then we always mean that all the arcs subtended by the sides of the polygon are also small, i.e. of small diameter.

We prove the existence of inscribed simple polygons all of whose sides are small as follows. Given a positive ϵ there is a positive δ such that any chord of the curve whose length is less than δ subtends an arc of the curve of diameter less than ϵ. Suppose that the sides of an inscribed polygon $C_1 C_2 ... C_n C_1$ are all less than δ, and that two sides $C_i C_{i+1}$ and $C_j C_{j+1}$ $(j > i+1)$ meet. Then at least one of the two chords $C_i C_j$, $C_{i+1} C_{j+1}$ is less than δ, say $C_i C_j$. Then for small δ only one of the two arcs $C_i C_{i+1} ... C_j$ and $C_j C_{j+1} ... C_i$ is small, say the first one. Then all the sides of the inscribed polygon $C_1 ... C_i C_j ... C_n C_1$ are less than δ, the arcs subtended by the sides have diameter less than ϵ, and the number of sides is less than n. If the sides of this polygon do not meet, then it is what we want; otherwise we apply the same argument as before to the new polygon and we shall arrive ultimately at a required polygon.

Our problem is to find the limits of the area of a simple polygon inscribed in a simple closed curve Γ as all the sides tend to zero. We shall need two lemmas. Given a plane set $\mathscr{E}$ and $\delta > 0$, the symbol $\{\delta, \mathscr{E}\}$ will denote the set of points of the plane distant less than δ from $\mathscr{E}$, but not belonging to $\mathscr{E}$. The Lebesgue two-dimensional measure of a set U is denoted by $|U|$.

LEMMA 1. *Given a bounded closed set $\mathscr{E}$, $|\{\delta, \mathscr{E}\}|$ is small for small δ.*

We shall want this lemma only for the case when $\mathscr{E}$ is a simple arc or a simple closed curve.

LEMMA 2. *A small chord of a simple closed curve divides it into two arcs, one of which is small.*

The two lemmas are well known and we have already used the second one.

Let Γ be a simple curve. Denote by G the domain interior to Γ and by $\bar{G}$ the closure of G, i.e. the set $G+\Gamma$. Similarly G', $\bar{G}'$; G'',..., are defined corresponding to the curves Γ', Γ'',.... Denote by $\Pi(\lambda)$ any simple polygon inscribed in Γ all of whose sides are of length less than λ.

THEOREM 1. $\limsup_{\lambda\to 0} |\Pi(\lambda)| \leqslant |\bar{G}|, \quad \liminf_{\lambda\to 0} |\Pi(\lambda)| \geqslant |G|.$

Proof. Given $\epsilon > 0$, let $\delta > 0$ be such that $|\{\delta, \Gamma\}| < \epsilon$. Further let $\lambda > 0$ be such that the smaller arc of Γ subtended by a chord of length less than λ is of diameter less than δ. The points, if any, of G that do not belong to $\Pi(\lambda)$ lie between sides of $\Pi(\lambda)$ and arcs subtended by these sides and consequently are all within δ from Γ. Hence the part of G belonging to $\Pi(\lambda)$ has area exceeding $|G|-\epsilon$. Thus $|\Pi(\lambda)| > |G|-\epsilon$. Similarly $|\Pi(\lambda)| < |\bar{G}|+\epsilon$.

COROLLARY. *If* $|\Gamma| = 0$, *then the limit of* $|\Pi(\lambda)|$, *as* $\lambda \to 0$, *exists and is equal to* $|G| = |\bar{G}|$.

Remark. When Γ contains linear segments, then $\Pi(\lambda)$ may have several consecutive sides lying on the same segment. The vertex between two such sides will be called a *nominal vertex*, and any other vertex a *genuine vertex*.

Suppose that Γ, always a simple closed curve, consists of the arcs

$$A_1 A_2,\ A_2 A_3, \dots,\ A_n A_1,$$

each of them convex outwards or each of them convex inwards. We say that Γ is *arc-wise convex* outwards or inwards respectively. Such curves are obviously of finite length. We call the points $A_1, A_2, \dots, A_n$ *the junctions.* I shall show that for any $\lambda > 0$ there exists a $\Pi(\lambda)$ such that none of its genuine vertices is an interior point of any segment belonging to Γ if such segments exist, and that all junctions are among its vertices. On every arc $A_i A_{i+1}$ (A_{n+1} being A_1) take a pair of points A_i'', A_{i+1}' so that $A_i, A_i'', A_{i+1}', A_{i+1}$ is a monotone sequence and the arcs $A_i A_i''$, $A_{i+1}' A_{i+1}$ are of length less than λ. If, for all i, the points A_i', A_i'' are sufficiently near A_i, then there exists a $\delta > 0$ such that for any i the distance between the arc $A_i'' A_{i+1}'$ and the sum-set of the arc $A_{i+1} \dots A_n \dots A_i$ and the chords $A_{i+1} A_{i+1}''$, $A_{i+2}' A_{i+2}$, $A_{i+2} A_{i+2}''$,..., $A_i' A_i$ exceeds δ. Let

$C_{i,1} = A_i'', C_{i,2}, \ldots, C_{i,k_i} = A_{i+1}'$ be a monotone sequence of points of the arc $A_i'' A_{i+1}'$ dividing it into sub-arcs of length less than $\min(\frac{1}{4}\delta, \lambda)$. Then the sequence of n sets of points $A_i, C_{i,1}, C_{i,2}, \ldots, C_{i,k_i}$ $(i = 1, 2, \ldots, n)$ form a monotone sequence on Γ, and the polygon $\Pi(\lambda)$ whose vertices are the points of this sequence is simple. If any of its genuine vertices are inner points of linear segments belonging to Γ, then add the end-points of all such intervals to the vertices of $\Pi(\lambda)$, and the new polygon will satisfy the required conditions.

For the general case, when $|\bar{G}| > |G|$, we shall construct an 'inner arcwise convex envelope' and an 'outer arcwise convex envelope' of Γ. Take $\delta > 0$ such that the diameter of one of the two arcs subtended by any chord of length not exceeding 4δ is less than $\frac{1}{10}d\Gamma$.† Denote by $C(\delta)$ the set of all circles of radius δ whose interior belongs to G and whose boundaries meet Γ in at least one point. An *open* arc of Γ of diameter less than $\frac{1}{10}d\Gamma$ whose end-points are on the same circle of $C(\delta)$ is called a *circumscribed arc*. If a pair of points of a circle of $C(\delta)$ belongs to Γ, then, by the definition of δ, one of the arcs of Γ between these points is a circumscribed arc. *The set of centres of circles of $C(\delta)$ coincides with the set of points of G distant δ from Γ.*

LEMMA 3. *If no circle of $C(\delta)$ passes through a point P of Γ, then P belongs to a circumscribed arc.*

Take on Γ points P_1, P_2 from two different sides of P such that‡

$$d\smile PP_1 = d\smile PP_2 = \tfrac{1}{10}d\Gamma.$$

Let E_1, E_2 be the domains of points of G distant less than δ from $\smile PP_1$, $\smile PP_2$ respectively, and F_1, F_2 be their outer boundaries (E_1, E_2 need not be simply connected). Further let F_1', F_2' be the components of $F_1 G$, $F_2 G$ respectively running from the arc PP_1 to the arc PP_2. Since F_1' has points inside F_2 and outside it, there is a common point O of F_1', F_2'. O is distant δ from each of the arcs PP_1, PP_2; let P_1', P_2' be points of PP_1, PP_2 respectively distant δ from O. Obviously $OP_1 > \delta$, for otherwise the arc $P_1 P P_2'$ would be subtended by a chord of length not exceeding 2δ, which is inconsistent with the fact that

$$d\smile P_1 P P_2' \geqslant d\smile P_1 P = \tfrac{1}{10}d\Gamma.$$

Similarly $OP_2 > \delta$ and $OP_3 > \delta$ for any P_3 on Γ outside the arc $P_1 P P_2$. Thus the circle $c(O, \delta)$ (of centre O and radius δ) belongs to $C(\delta)$ and passes

† d stands for 'diameter'. ‡ $\smile$ stands for 'arc'.

through P_1', P_2'. Since P is not on $c(O, \delta)$, each of the points P_1', P_2' is different from P, and thus P is an interior point of the circumscribed arc $P_1' P P_2'$.

LEMMA 4. *If a circumscribed arc $P_1 P P_2$ subtends an arc of angle exceeding π of a circle of $C(\delta)$, then either it subtends an arc not exceeding π of another circle of $C(\delta)$ or it is a true part of another circumscribed arc.*

We take the arcs $P_2 P_3$ and $P_4 P_1$ of diameter $\frac{1}{10} d\Gamma$ outside $P_1 P P_2$ and consider the inner domains E_1, E_2 of points of G distant less than δ from $P_4 P_1$, $P_2 P_3$ respectively. Then by an argument similar to one employed in Lemma 3 we arrive at the proof.

It is easy to see that, if two circumscribed arcs overlap partly: that is, so that no one of them is included in the other one, then they are circumscribed about the same circle of $C(\delta)$, and thus their sum is itself a circumscribed arc. From this it follows that, if the arc $P_1 P_2$ is the sum of a set of circumscribed arcs, then $P_1 P_2$ is itself a circumscribed arc. For obviously $P_1 P_2$ is an open arc. Let $P_1' P_2'$ be any closed arc included in $P_1 P_2$. Then by the Heine–Borel theorem it can be covered by a finite sequence of partly overlapping circumscribed arcs belonging to $P_1 P_2$. Hence $P_1' P_2'$ is included in a single circumscribed arc which itself is included in $P_1 P_2$. We arrive finally at the required conclusion by a limiting argument. Write now $\Gamma = \Delta_1 + \Delta_2$, where Δ_1 is the sum of all circumscribed arcs and Δ_2 the rest of Γ. The set Δ_1, being the sum of open sets on Γ, is itself an open set and can be represented as the sum of disjoint open arcs of Γ. By what has just been shown, each of these arcs is itself a circumscribed arc. Thus Γ *is represented as the sum Δ_1 of disjoint circumscribed arcs each subtended by an arc of a circle of $C(\delta)$ of angle not exceeding π, and the set Δ_2 through each point of which passes a circle of $C(\delta)$.*

We shall now construct an inner arc-wise convex envelope of Γ. Define $\eta > 0$ so that the chord of any arc of Γ of diameter not less than δ and less than $\frac{1}{2} d\Gamma$ exceeds 4η. Obviously $\eta < \frac{1}{4}\delta$.

We shall now divide Γ into a finite number of sub-arcs. First, all the arcs of Δ_1 of diameter not less than η are taken for such sub-arcs, and then the rest of Γ is divided into sub-arcs of diameter less than η by points of Δ_2. Suppose that the subdivision points form a monotone sequence $A_1, A_2, \ldots, A_n, A_{n+1} = A_1$. From every point A_i draw the radius $A_i B_i$ of a circle of $C(\delta)$ passing through A_i; in the case when $A_i A_{i+1}$ is a circumscribed arc the radii from A_i and A_{i+1} are those of the circle whose arc of angle not exceeding π is subtended by $A_i A_{i+1}$,

so that B_i, B_{i+1} coincide. If $A_{i-1}A_i$ and A_iA_{i+1} are both circumscribed arcs, then from A_i two different radii A_iB_i', A_iB_i'' are drawn. A pair of radii A_iB_i, A_jB_j $(i<j)$ do not meet unless $j=i+1$ and A_iA_{i+1} is a circumscribed arc, for otherwise at least one of the inequalities $A_iB_j<\delta$, $A_jB_i<\delta$ would be true, which would mean that at least one of the points A_i, A_j is inside a circle of $C(\delta)$. We shall now define the *convex hull* of the arc A_iA_{i+1} from inside Γ. If A_iA_{i+1} is not a circumscribed arc, then consider the simple arc $B_iA_iA_{i+1}B_{i+1}$ formed by the radius B_iA_i, the arc A_iA_{i+1}, and the radius $A_{i+1}B_{i+1}$. The set

$$\{h, \smile A_iA_{i+1}\} \quad (0<h<\delta)$$

of points of the plane whose distance from the arc A_iA_{i+1} is less than h is divided by $B_iA_iA_{i+1}B_{i+1}$ into two domains. One of them contains points of G in any neighbourhood of every interior point of $\smile A_iA_{i+1}$, and the other does not; denote by $(h, \smile A_iA_{i+1})$ the first one. For $h<4\eta$ we have $(h, A_iA_{i+1})\subset G$. For suppose a point A' of Γ belongs to (h, A_iA_{i+1}) and let A'' be the point of A_iA_{i+1} nearest to A'. Since the radii A_iB_i, $A_{i+1}B_{i+1}$ are met by Γ only at points A_i, A_{i+1}, each of the two arcs into which Γ is divided by points $A'A''$ embraces one of the radii A_iB_i, $A_{i+1}B_{i+1}$, and thus the diameter of each of them exceeds δ. Hence the chord $A'A''>4\eta$, which is a contradiction. An immediate corollary is that, when $h<2\eta$, the sets $(h, \smile A_iA_{i+1})$, $(h, \smile A_jA_{j+1})$ *have no points in common for* $i\neq j$. Consider the set consisting of the arc A_iA_{i+1} and of the chords joining any pair of points of the arc A_iA_{i+1} and totally belonging (the chords) to G. The boundary of this set from the inner side of $B_iA_iA_{i+1}B_{i+1}$ *is the convex hull of* A_iA_{i+1}. If the arc A_iA_{i+1} is itself convex inwards, then it coincides with its hull. If the chord A_iA_{i+1} is included in G, then it is the convex hull. If neither of these two cases takes place, then the convex hull consists of some straight segments and of some points of the arc A_iA_{i+1}. In all cases A_i, A_{i+1} are the end-points of the hull. Since $d\smile A_iA_{i+1}<\eta$ and $\smile A_iA_{i+1}$ is outside $c(B_i,\delta)$, we have, for any point A of the arc A_iA_{i+1},

$$\text{angle}(A_iB_i, A_iA) > \arccos\frac{\eta}{2\delta} > \arccos\tfrac{1}{8}.$$

Thus the inner one-sided tangents of the convex hull at the points A_i, A_{i+1} form angles exceeding $\arccos\frac{1}{8}$ with the radii A_iB_i, $A_{i+1}B_{i+1}$ respectively. Obviously the convex hull of A_iA_{i+1} is included in $(\eta, \smile A_iA_{i+1})$ and consequently convex hulls of any pair of arcs that are not circumscribed arcs have no common interior points and, if the arcs are not adjacent, no common points at all.

If $A_i A_{i+1}$ is a circumscribed arc, then the chord $A_i A_{i+1}$ is its convex hull. Thus convex hulls of all arcs $A_i A_{i+1}$ have been defined: they are all convex inwards. The convex hulls of all the arcs $A_1 A_2, A_2 A_3, \ldots, A_n A_1$ form a simple arcwise convex (inwards) curve Γ' called *an inner δ-envelope* of Γ. We have $G' \subset G$. All the junctions $A_1, \ldots, A_n$ are common points of Γ and Γ' and so are all the points of Γ' that are not inner points of straight segments possibly belonging to convex hulls of arcs

$$A_1 A_2, \ldots, A_n A_1.$$

We see that Γ' is obtained from Γ by replacing certain sub-arcs of Γ by their chords which totally belong to G and are of length not exceeding 2δ. Γ' is always of finite length. By Lemma 1 and Lemma 2, $|G'| \to |G|$, as $\delta \to 0$. Similarly an outer δ-envelope Γ'' of Γ is defined as an arcwise convex (outwards) simple curve which satisfies the condition $G'' \supset G$ and is obtained from Γ by replacing some of its sub-arcs by their chords which are totally outside G and are of length not exceeding 2δ. Γ'' is of finite length and $|\bar{G}''| = |G''| \to |\bar{G}|$, as $\delta \to 0$.

Given $\epsilon > 0$, choose $\delta > 0$ so that

$$|G - G'| < \tfrac{1}{2}\epsilon, \qquad |\bar{G}'' - \bar{G}| < \tfrac{1}{2}\epsilon$$

for an inner δ-envelope Γ' and an outer δ-envelope Γ''. It was shown before that for any $\lambda > 0$ simple polygons $\Pi'(\lambda)$, $\Pi''(\lambda)$ can be inscribed into Γ', Γ'' respectively so that none of their genuine vertices is an interior point of a linear segment belonging to Γ', Γ''. For λ small enough

$$|\Pi'(\lambda)| < |G'| + \tfrac{1}{2}\epsilon, \qquad |\Pi''(\lambda)| > |\bar{G}''| - \tfrac{1}{2}\epsilon.$$

Hence
$$|\Pi'(\lambda)| < |G| + \epsilon, \qquad |\Pi''(\lambda)| > |\bar{G}| - \epsilon.$$

But all the genuine vertices of $\Pi'(\lambda)$, $\Pi''(\lambda)$ are on Γ. Thus $\Pi'(\lambda)$, $\Pi''(\lambda)$ are inscribed in Γ too, and their sides are of length not exceeding 2δ. Thus we have that for a polygon $\Pi(\lambda)$ inscribed in Γ

$$\liminf_{\lambda \to 0} |\Pi(\lambda)| \leqslant |G|, \qquad \limsup_{\lambda \to 0} |\Pi(\lambda)| \geqslant |\bar{G}|,$$

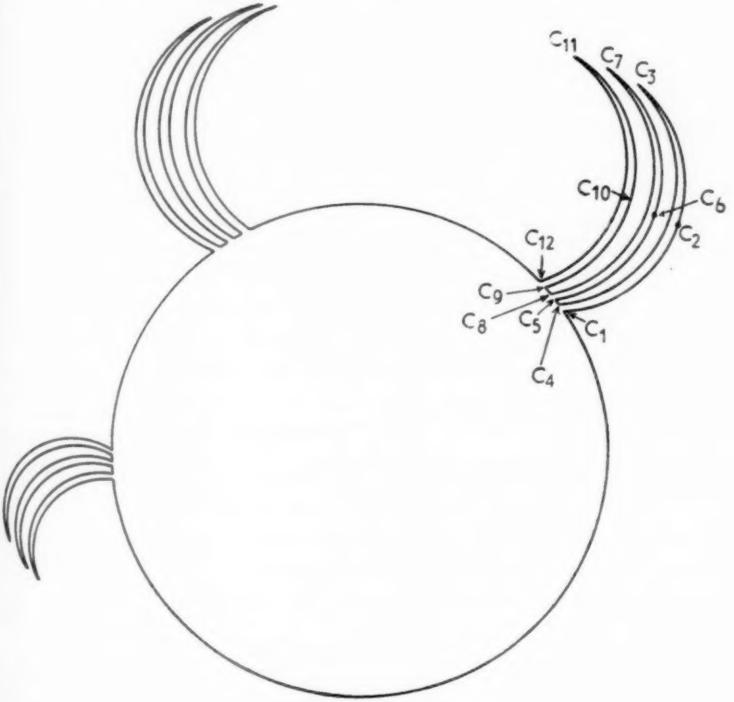
which together with Theorem 1 leads to

THEOREM 2. $\liminf |\Pi(\lambda)| = |G|$ *and* $\limsup |\Pi(\lambda)| = |\bar{G}|$.

THEOREM 3. *Any number* $|G| + \alpha$ $(0 < \alpha < |\bar{G}| - |G|)$ *is a limit value of the area of an inscribed simple polygon.*

The idea of the proof is this. Divide Γ into two arcs ABC and CDA so that $|\smile ABC| = \alpha$ and form Γ_0 of the outer arcwise convex envelope of the arc $A'BC'$ included in ABC and nearly equal to it, of the inner envelope

of the arc $C''DA''$ included in CDA and nearly equal to it, and of the chords $A'A''$, $B'B''$. For a proper choice of the points A', A'', C', C'', Γ_0 is a simple curve. Then we find a polygon $\Pi_0(\lambda)$ inscribed in both Γ_0 and Γ and of area nearly equal to $G+\alpha$.



In conclusion, consider limits of the area of polygons $C_1 C_2 ... C_n C_1$ inscribed in a simple curve Γ, where $C_1, C_2, ..., C_n$ is a monotone sequence on Γ but the polygons are not restricted by the condition to be simple. The area of such polygons will be defined by the value of the integral $\frac{1}{2}\int (x\,dy - y\,dx)$ taken along the boundary of the polygon in the direction of increasing suffixes of vertices assuming that this direction coincides with the positive direction on Γ. *Then there are simple curves for which all the real numbers from $-\infty$ to $+\infty$ are limits of the area of an inscribed polygon.* That can be seen from the following phenomenon represented on the diagram. Suppose that Γ has a large number of thin horns of considerable convexity very close to each other and 'almost parallel',

such as $C_1 C_2 C_3 C_4$, $C_5 \ldots C_8, \ldots$ and that the points $C_1, C_2, \ldots$ are consecutive vertices of an inscribed polygon. This polygon is not simple and the integral $\frac{1}{2}\int (x\,dy - y\,dx)$ along sets of three consecutive sides $C_1 C_2 C_3 C_4$, $C_5 C_6 C_7 C_8, \ldots$ is approximately equal to the area of the triangle $C_1 C_2 C_3 C_1$, so that, if there are many such horns, their contribution to the area will be large. If we have sets of such horns of decreasing diameters in sufficient numbers and also sets of horns with convexity directed to the opposite side, then there are inscribed polygons of sides as small as we please whose area is any given real number.

A NOTE ON THE VARIETY OF GHERARDELLI

By A. ZOBEL (*London*)

[Received 21 January 1955]

1. In this note we shall be concerned with the irreducible non-singular variety W whose points represent unexceptionally the ∞^4 second-order curve elements (E_2) in the plane as defined by E. Study (**1**). W was first constructed by G. Gherardelli (**2**), and again from a somewhat different point of view by J. G. Semple (**3**). Algebraic bases for curves and threefolds, but not for surfaces, on W were found by Gherardelli [(**2**), 824]; I propose here to employ the method of degenerate collineations, in the manner of B. van der Waerden (**4**) and W. V. D. Hodge and D. Pedoe [(**5**), 141 ff., 197 ff., 337 ff.], to obtain a base also for surfaces on W.

I am much indebted to Professor J. G. Semple, whose lecture to the 1954 British Mathematical Colloquium provided the stimulus for this investigation, and to Dr. D. B. Scott, whose constructive criticism assisted greatly in its completion.

2. The following results, relevant to our purpose, are due to the first three authors quoted above and to F. Severi [(**6**), § 59]:

(i) The line elements (E_1) of the plane can be represented by a non-singular threefold U on which algebraic bases for curves and surfaces are formed respectively by the pairs g_x, g_y and G_x, G_y, where

g_x represents the E_1 with fixed origin,
g_y represents the E_1 with fixed tangent,
G_x represents the E_1 whose tangents pass through a fixed point,
G_y represents the E_1 whose origins lie on a fixed line.

(ii) On W, the threefolds C, representing the cuspidal E_2, and F, representing the inflexional E_2, are each a model of U. Hence an algebraic base for surfaces on F is formed by Q and L, defined in terms of inflexional E_2 as G_x and G_y are, respectively, in terms of E_1.

(iii) W consists of ∞^3 lines t joining points on C and F whose (respectively cuspidal and inflexional) associated E_2 have the same origin and tangent; the points of a line t represent all E_2 with a given origin and tangent: that is, all E_2 based on a given E_1. The aggregate of the t is thus birationally equivalent to U. In the correspondence between U and W relating each point of U to the appropriate t, g_x and g_y on U

correspond respectively to P, representing the E_2 with given origin, and R, representing the E_2 with given tangent, on W. Hence it follows from (i) above that P and R form an algebraic base for surfaces on W composed of lines t.

(iv) The coordinates in the space of W can be, as a matter of convenience, divided into two sets $\mathbf{X}$ and $\mathbf{Y}$ which can be parametrized on W in the form

$$\left.\begin{aligned} X_{abcde} &= rx_a x_b x_c x_d y_e \\ Y_{jmnpq} &= sy_j y_m y_n y_p x_q \end{aligned}\right\} \quad (a, b, c, d, e, j, m, n, p, q = 0, 1, 2),$$
$$\mathbf{x}.\mathbf{y} = 0,$$

where $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$ are respectively the origin and tangent of the E_2 represented, and $r:s$ is a position parameter on the appropriate t such that $r = 0$ on F and $s = 0$ on C. The pair $(\mathbf{X}, \mathbf{Y})$ forms a single set of homogeneous coordinates which is also, on W, separately homogeneous in (r, s) and in the pair $(\mathbf{x}, \mathbf{y})$, but not in $\mathbf{x}$ or $\mathbf{y}$ alone; in this sense the parametrization is improper.†

3. If $(\mathbf{X}', \mathbf{Y}')$ is a point of W, the point $(k\mathbf{X}', \mathbf{Y}')$ also lies on W, whatever the value of k; in fact, unless $\mathbf{X}' = \mathbf{0}$ or $\mathbf{Y}' = \mathbf{0}$, the totality of such points forms the line t through $(\mathbf{X}', \mathbf{Y}')$. Consequently the equations

$$g_j(\mathbf{X}, \mathbf{Y}) = 0 \quad (j = 1, 2, \ldots)$$

of W are homogeneous in $\mathbf{X}$ and $\mathbf{Y}$ separately. Thus a collineation $A(k)$ of the form

$$\mathbf{X}^* = k\mathbf{X}, \qquad \mathbf{Y}^* = \mathbf{Y}$$

leaves W invariant unless $k = 0$ or ∞, i.e. provided that $A(k)$ is non-singular.

If $(\mathbf{X}', \mathbf{Y}')$ satisfies the (homogeneous algebraic) equation $f(\mathbf{X}, \mathbf{Y}) = 0$, then $(k\mathbf{X}', \mathbf{Y}')$ satisfies $f(\mathbf{X}, k\mathbf{Y}) = 0$. Thus a variety S on W with equations

$$g_j(\mathbf{X}, \mathbf{Y}) = f_i(\mathbf{X}, \mathbf{Y}) = 0 \quad (i = 1, 2, \ldots)$$

is, for $k \neq 0$ or ∞, transformed by $A(k)$ into a variety $S(k)$, also lying on W and given by

$$g_j(\mathbf{X}, \mathbf{Y}) = f_i(\mathbf{X}, k\mathbf{Y}) = 0.$$

Suppose now that S is an irreducible surface not lying on C, and let K be a straight line with generic point $(\kappa, 1)$. Since $S = S(1)$ is irreducible, so is $S(\kappa)$ with generic point, say, $\boldsymbol{\zeta} = (\boldsymbol{\Xi}, \mathbf{H})$. Let T be the correspondence between W and K determined by the pair $(\boldsymbol{\zeta}, \kappa)$; since

† A full account of the derivation and properties of this parametrization is given by J. G. Semple [(**3**), 29–31].

T has a generic point, it is necessarily irreducible. By definition of $S(\kappa)$, $(\boldsymbol{\zeta}, \kappa)$ satisfies

$$g_j(\boldsymbol{\Xi}, \mathbf{H}) = f_i(\boldsymbol{\Xi}, \kappa\mathbf{H}) = 0;$$

hence, if $\mathbf{z} = (\mathbf{X}, \mathbf{Y})$, any proper specialization $(\mathbf{z}, k)$ of $(\boldsymbol{\zeta}, \kappa)$ satisfies

$$g_j(\mathbf{X}, \mathbf{Y}) = f_i(\mathbf{X}, k\mathbf{Y}) = 0.$$

In particular, if $S'(0)$ is the variety of points $\mathbf{z}$ such that $(\mathbf{z}, 0)$ is a proper specialization of $(\boldsymbol{\zeta}, \kappa)$, i.e. the variety on W corresponding under T to the specialization $\kappa \to 0$, then $S'(0)$ satisfies the equations

$$g_j(\mathbf{X}, \mathbf{Y}) = f_i(\mathbf{X}, \mathbf{0}) = 0.$$

Also, following an argument due to Hodge and Pedoe [(**5**) Ch. XI, § 6], the Cayley image of $S(\kappa)$ is a generic point, with coordinates $\boldsymbol{\tau}$ which are by definition of $S(\kappa)$ rational functions of κ, of a curve J whose points are given by the specializations of κ. Thus in the correspondence between J and K defined by the pair $(\boldsymbol{\tau}, \kappa)$ a unique point of J corresponds to every point of K, and in particular to the point $(0, 1)$ of K. For $k \neq 0$ or ∞, the point of J corresponding to $(k, 1)$ on K is the Cayley image of the surface $S(k)$, and in particular $k = 1$ gives rise to the Cayley image of S; we write $S(0)$ for the surface whose Cayley image is the point of J corresponding to $k = 0$.

Then owing to a theorem proved by Hodge and Pedoe [(**5**) 115, Th. II] $S(0) = S'(0)$; thus $S(0)$ is a surface on W, equivalent there to S, and satisfying

$$g_j(\mathbf{X}, \mathbf{Y}) = f_i(\mathbf{X}, \mathbf{0}) = 0.$$

4. Suppose now that $\mathbf{X}' \neq \mathbf{0}$, and that

$$g_j(\mathbf{X}', \mathbf{Y}') = f_i(\mathbf{X}', \mathbf{0}) = 0;$$

then, taking into account the separate homogeneity of the g_j in $\mathbf{X}$ and $\mathbf{Y}$, we see that the point $(\mathbf{X}', \mathbf{0})$ satisfies

$$g_j(\mathbf{X}, \mathbf{Y}) = f_i(\mathbf{X}, \mathbf{Y}) = 0.$$

Thus, if $(\mathbf{X}', \mathbf{Y}')$ lies on $S(0)$ but not on F, then $(\mathbf{X}', \mathbf{0})$ lies on S. Since $(\mathbf{X}', \mathbf{0})$ is the point of C on the line t passing through $(\mathbf{X}', \mathbf{Y}')$, it follows that all points of $S(0)$ not on F lie on the ∞^1 lines t meeting the intersection curve of S and C; hence and since $S(0)$ is a surface, any component of $S(0)$ not on F is composed of lines t. $S(0)$ being equivalent to S, it follows now from (ii), (iii) of § 2 that S can be expressed, by means of an algebraic equivalence on W, in terms of P, R, Q, L.

The only restriction on S, apart from its irreducibility, was that it should not lie on C. Suppose then that S does lie on C; and let S', $S+S''$ be the respective intersections of W with two pairs of primals of order n

such that each member of the second pair passes through S. Then, since W is non-singular, S' and S'' are irreducible and do not lie on C if n is taken large enough and the primals are chosen with sufficient generality. Thus, even if S does lie on C, it is equivalent on W to the difference of two irreducible surfaces neither of which lies on C; and taking into account the previous result we conclude that

P, R, Q, L form an algebraic base for surfaces on W.

5. It is easy to show by means of the intersection determinant that the base just found is minimal in Severi's sense [(**6**), § 6]. For, if (V, V') is the virtual intersection number of V and V' on W, and v is used to denote purely virtual numbers, then

$$\begin{vmatrix} (P,P) & (P,R) & (P,Q) & (P,L) \\ (R,P) & (R,R) & (R,Q) & (R,L) \\ (Q,P) & (Q,R) & (Q,Q) & (Q,L) \\ (L,P) & (L,R) & (L,Q) & (L,L) \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & v & v \\ 0 & 1 & v & v \end{vmatrix} = 1,$$

which proves the result. The numbers given represent actual intersections, and can thus be verified without difficulty. It can also be shown that $(Q, Q) = -3$, $(Q, L) = 0$, $(L, L) = 3$; but, since these results are not required for our purpose, the rather lengthy proof would be out of place in the present context.

It should finally be mentioned that, as an alternative to Gherardelli's approach, the method employed here could be used equally well to find the bases for threefolds and curves on W.

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VAN DER CORPUT'S METHOD AND THE THEORY OF EXPONENT PAIRS

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1. THE results given in this paper were obtained in 1945† during the course of some work done on van der Corput's method. Since there has been a revival of interest in the method recently and since the results I give are easily applicable to a variety of problems in the theory of numbers, it may be of interest to put them on record. The proofs are somewhat long and involve much rather heavy algebra; for these reasons they are omitted.

Van der Corput's method of approximating to trigonometric sums makes use of two processes A and B. Each process replaces a given trigonometric sum

$$S_1 = \sum_{a \leqslant n < b} e^{2\pi i f(n)}$$

by another sum S_2 involving a different function. In effect, process A consists of a suitably chosen application of Schwarz's inequality, while process B corresponds to an application of the Poisson summation-formula and yields an approximate functional equation for S_1 in terms of S_2. Process B when applied twice transforms S_2 back into S_1, but process A can be applied as many times as desired and yields a different sum at each application.

Many of the estimates of error terms in problems in the theory of numbers arise from a suitably chosen sequence of applications of the two processes to a certain trigonometric sum. In work of this type the actual sequence chosen by the author may seem haphazard, and the reader may suspect that a different sequence might give a sharper estimate. I show that this is, in fact, the case and that any estimate obtainable by a finite number of applications of the processes A and B can be improved by choice of a different sequence. On the other hand, there does exist, as a limiting case, a best possible estimate for the method which cannot be improved upon.

Van der Corput's theory of *exponent pairs* (**15**), or *exponent systems*, as originally developed, replaced a given exponent pair by a number of

† With the exception of V and VI of § 4, which relate to more recent applications of the method.

others so that, at each application of the processes A and B, the number of exponent pairs increased and the predominating exponent pair might not be discovered until the final stages of the investigation. Phillips (**5**) introduced a great simplification in the theory, without lessening the power of the method, by showing that only one exponent pair need be retained at each stage. For this reason I use his form of the theory. It may be noted here that Phillips's process A^q, obtained by iterating A q times, is more powerful than van der Corput's process A_q (which is not an iteration) for $q > 1$, the two processes being identical for $q = 1$. This means that there is no advantage in using A_q rather than A^q.

A more recent form of the method is due to Titchmarsh (**11**, **12**, **13**) and makes use of two-dimensional sums. In the problems where it has been applied, Titchmarsh's method provides sharper estimates than those given by the results obtained here. However, the method has not yet been formulated into a readily applicable theory of exponent triples, so that each problem has to be considered on its own merits, and the special properties of the functions f occurring in the sums at each application of the various processes have to be considered.

2. Definitions and notation

A pair (k, l) of real numbers is called an *exponent pair* if

$$0 \leqslant k \leqslant \tfrac{1}{2}, \qquad \tfrac{1}{2} \leqslant l \leqslant 1,$$

and if, corresponding to every positive number s, there exist two numbers r and c depending only on s (r an integer greater than 4 and $0 < c < \frac{1}{2}$) such that the inequality

$$\sum_{a \leqslant n < b} e^{2\pi i f(n)} = O(z^k a^l)$$

holds with respect to† s and u when the following conditions are satisfied:

$$u > 0, \qquad 1 \leqslant a < b < au, \qquad y > 0, \qquad z = ya^{-s} > 1,$$

$f(n)$ being any real function with differential coefficients of the first r orders in the interval $a \leqslant n \leqslant b$ (a, b are integers), and

$$|f^{(p+1)}(n) - (-1)^p ys(s+1)...(s+p-1)n^{-s-p}| < cys(s+1)...(s+p-1)n^{-s-p}$$

for $a \leqslant n \leqslant b$, $0 \leqslant p \leqslant r-1$.

It follows immediately that $(0, 1)$ is an exponent pair since

$$\left|\sum_{a \leqslant n < b} e^{2\pi i f(n)}\right| \leqslant b-a < au = uz^0 a.$$

It may be noted that z is effectively $f'(a)$.

† i.e. the constant implied by the O notation depends only on s and u.

By an application of the A-process Phillips (**5**) showed that, if (κ,λ) is an exponent pair, so is (k,l), where

$$k = \frac{\kappa}{2(1+\kappa)}, \qquad l = \frac{1}{2}+\frac{\lambda}{2(1+\kappa)}.$$

By applying the B-process he showed that, if (κ,λ) is an exponent pair and if $2\lambda+\kappa \geqslant \frac{3}{2}$, then so is (k,l), where

$$k = \lambda-\tfrac{1}{2}, \qquad l = \kappa+\tfrac{1}{2}.$$

We write these results as

$$\left(\frac{\kappa}{2(1+\kappa)}, \frac{1}{2}+\frac{\lambda}{2(1+\kappa)}\right) = A(\kappa,\lambda), \qquad (\lambda-\tfrac{1}{2},\kappa+\tfrac{1}{2}) = B(\kappa,\lambda).$$

The operators A and B always act towards the right. Iterates of A we denote by A^2, A^3, etc. The restriction $2\lambda+\kappa \geqslant \frac{3}{2}$ is unimportant as it is always satisfied by exponent pairs to which the process B is applied.

In several problems to which van der Corput's method can be applied, the best estimates of error terms arise when an exponent pair (k,l) is chosen for which $k+l$ is as small as possible. Thus, in the circle and divisor problems, the exponent in the error term is $k+l-\frac{1}{2}$ and the familiar indices $\frac{1}{3}$, $\frac{33}{100}$ (actually $\frac{163}{494}$), $\frac{27}{82}$, and $\frac{229}{696}$ arise from the sequences

$A\,B$,

$A\,B\,A_3\,B\,A\,B\,A\,B$ [van der Corput† (**15**)],

$A\,B\,A_4'\,B$ [Nieland (**4**), Titchmarsh† (**10, II**)],

$A\,B\,A^3\,B\,A^2\,B\,A^2\,B$ [Phillips (**5**)].

3. Statement of results

We regard each exponent pair as a point (k,l) in two-dimensional Euclidean space and investigate the set of all points obtainable by applying the processes A and B to $(0,1)$. We make use of a further 'convexity' process C which asserts that, if (k_1,l_1) and (k_2,l_2) are exponent pairs, so is (κ,λ), where

$$\kappa = tk_1+(1-t)k_2, \qquad \lambda = tl_1+(1-t)l_2,$$

and t is any number satisfying $0 \leqslant t \leqslant 1$. The proof of this is obvious.

† The operator A_3 is van der Corput's variant mentioned earlier. If A^3 is substituted, a lower index $\frac{77}{234}$ arises. Similarly, Titchmarsh's operator A_4' is less powerful than A^4, but in this particular sequence gives a lower index than A^4 would give.

Let S_0 denote the set of all exponent pairs (k, l) which can be obtained from $(0, 1)$ by use of a finite number of the processes A, B, and C. Then S_0 contains the straight line segment Γ_0 joining $(0, 1)$ to $(\frac{1}{2}, \frac{1}{2}) = B(0, 1)$. The set of those points of S_0 which do not lie on Γ_0 we denote by S. Then

(i) S is a convex, open set of points.

(ii) S is symmetrical about the line L whose equation is

$$y = x+\tfrac{1}{2}.$$

(iii) the boundary of S consists of Γ_0 and a curve Γ which joins $(0, 1)$ to $(\frac{1}{2}, \frac{1}{2})$ and lies in the triangle formed by the lines Γ_0, $x = 0$, $y = \frac{1}{2}$;

(iv) the curve Γ is composed partly of segments of straight lines. One such segment, σ say, crosses the line L and contains the two points Q (κ_0, λ_0) and Q^* $(\lambda_0-\frac{1}{2}, \kappa_0+\frac{1}{2})$ which are images in L $(\lambda_0 > \kappa_0+\frac{1}{2})$. Here (κ_0, λ_0) is the limit of a sequence of operations A, B applied to $(0, 1)$, or, what is the same thing, to $(\frac{1}{6}, \frac{2}{3}) = AB(0, 1)$ or any other point of S_0. In fact,

$$(\kappa_0, \lambda_0)$$
$$= ABA^2.ABA.AB.AB.ABA.AB.AB.ABA.ABA.AB....(\tfrac{1}{6}, \tfrac{2}{3}).$$

Here the operations have been separated into groups of the form ABA^r, as in evaluating the limit a method was developed which, at each stage, replaced an operator P_n by $P_{n+1} = P_n\,ABA^r$; here $r = 0$, 1, or 2, and can be determined uniquely at each stage. If we stop at the tenth stage, we obtain

$$\begin{aligned}(\kappa_{10}, \lambda_{10}) &\\ &= ABA^2.ABA.AB.AB.ABA.AB.AB.ABA.ABA.AB(\tfrac{1}{6}, \tfrac{2}{3})\\ &= (\tfrac{141841}{1019718}, \tfrac{703527}{1019718}). \qquad (1)\end{aligned}$$

(v) The segment σ, which is part of Γ, forms part of the line

$$x+y = \tfrac{1}{2}+\alpha, \qquad (2)$$

where
$$\alpha = 0{\cdot}32902\,13568\ldots. \qquad (3)$$

More precisely,

$$0{\cdot}32902\,13568\,4 \leqslant \alpha < 0{\cdot}32902\,13568\,8\ldots = \kappa_{10}+\lambda_{10}-\tfrac{1}{2}. \qquad (4)$$

The line (2) is a tac-line to S.

(vi) In particular,

$$\inf_{(k,l)\in S} (k+l) = \tfrac{1}{2}+\alpha, \qquad (5)$$

and, for any $\epsilon > 0$, $(\frac{1}{2}\alpha+\epsilon, \frac{1}{2}\alpha+\frac{1}{2}+\epsilon)$ belongs to S and is an exponent pair.

Finally, we note that, where above we have stated that a method, M say, is not so powerful as that used here, we mean that the set of exponent pairs (k, l) obtainable by M forms a subset of S_0.

4. Applications

The results stated in § 3 show that a limiting process can be applied to van der Corput's method and in this way estimates can be obtained which are not improvable by choosing different combinations of the two processes while retaining the essential features of the one-dimensional method.

I mention here various problems to which the results can be applied. The symbol ϵ denotes any positive constant.

(I) *The circle problem.* The method gives

$$\sum_{m^2+n^2 \leqslant x} 1 = \pi x + O(x^{\alpha+\epsilon}),$$

which is inferior to Hua's result (**2**) obtained by the two-dimensional method,

$$\sum_{m^2+n^2 \leqslant x} 1 = \pi x + O(x^{13/40} \log^{9/8} x).$$

(II) *The divisor problem.* The method gives

$$\sum_{mn \leqslant x} 1 = x \log x + (2\gamma - 1)x + O(x^{\alpha+\epsilon}),$$

which is inferior to Richert's result (**7**) obtained by the two-dimensional method,

$$\sum_{mn \leqslant x} 1 = x \log x + (2\gamma - 1)x + O(x^{15/46} \log^{30/23} x),$$

but superior to the previous best estimate due to Nieland (**4**).

(III) *The order of* $\zeta(\frac{1}{2}+it)$. The method gives

$$\zeta(\tfrac{1}{2}+it) = O(|t|^{\frac{1}{2}\alpha+\epsilon})$$

for large t. This is inferior to S. H. Min's result (**3**)

$$\zeta(\tfrac{1}{2}+it) = O(|t|^{15/92+\epsilon}).$$

(IV) *Piltz's divisor problem for* $k = 3$

If $d_3(n)$ denotes the number of ways in which n can be expressed as a product of three factors, then

$$\sum_{n \leqslant x} d_3(n) = x f_3(\log x) + \Delta_3(x),$$

where $f_3(\log x)$ is a quadratic polynomial in $\log x$, being the residue of $\zeta^3(s)x^{s-1}/s$ at $s = 1$. Atkinson (**1**) has shown that

$$\Delta_3(x) = O(x^{37/75+\epsilon}).$$

It is easy to show that his index $\frac{37}{75}$ can be replaced by

$$\beta = \frac{2l+k+1}{3(l+1)},$$

where (k,l) is an exponent pair, provided that $10l+19k > 9$. For $(k,l) = A(\frac{1}{6}, \frac{2}{3})$, $\beta = \frac{37}{75}$; however, if we take $(k,l) = A\left(\frac{\alpha}{2}+\epsilon', \frac{\alpha+1}{2}+\epsilon'\right)$, we get the better estimate

$$\beta = \frac{7\alpha+10}{3(4\alpha+7)}+\epsilon = 0{\cdot}49314\ 66...+\epsilon.$$

This can probably be reduced still further by choosing β so that the straight line $\beta = \dfrac{2l+k+1}{3(l+1)}$ is a tac-line to the curve Γ.

(V) *The number of abelian groups of a given order.* If $a(n)$ denotes the number of different abelian groups of order n, then Richert (**6**) has shown that

$$A(x) = \sum_{n \leqslant x} a(n) = c_1 x + c_2 x^{\frac{1}{2}} + c_3 x^{\frac{1}{3}} + O(x^{3/10} \log^{9/10} x).$$

Here the exponent $\frac{3}{10}$ arises as $(k+1)/(k+4)$, where (k,l) is an exponent pair such that $l = 2k$, and for the choice

$$(k,l) = (\tfrac{2}{7}, \tfrac{4}{7}) = BA^2B(0,1). \tag{6}$$

We can now replace this by

$$\left(\frac{\alpha+1}{2(\alpha+2)}+\epsilon'', \frac{\alpha+1}{\alpha+2}+2\epsilon''\right) = BA(\tfrac{1}{2}\alpha+\epsilon', \tfrac{1}{2}\alpha+\tfrac{1}{2}+\epsilon') \tag{7}$$

and so replace the error term $O(x^{3/10} \log^{9/10} x)$

by $$O(x^{\beta+\epsilon}),$$

where $$\beta = \frac{3\alpha+5}{9\alpha+17} = 0{\cdot}29993\,519...$$

(VI) *The distribution of square-free numbers.* If q_n is the nth square-free number, K. F. Roth (**9**) has shown that

$$q_{n+1}-q_n = O\{n^{3/13}(\log n)^{4/13}\}$$

by using a result of van der Corput's [(**17**) (II), Satz 3]. By using the theory of exponent pairs, Richert (**8**) has improved this to

$$q_{n+1}-q_n = O(n^{2/9} \log n). \tag{8}$$

Actually Richert's method shows that

$$q_{n+1}-q_n = O(n^{\rho} \log n), \tag{9}$$

where $\rho = k/(k+1)$, and (k, l) is an exponent pair such that $l = 2k$. By taking the exponent pair (6) he obtains his result (8). If the exponent pair (7) is used instead of (6) in (9), we get the slightly better estimate

$$q_{n+1} - q_n = O(n^{\gamma+\epsilon}),$$

where
$$\gamma = \frac{\alpha+1}{3\alpha+5} = 0{\cdot}22198\,215\ldots.$$

The results of § 3 can also be applied to estimate $\zeta(\sigma+it)$ for $\frac{1}{2} < \sigma < 1$ and to various mean-value problems for the zeta-function.

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THE FUNDAMENTAL GROUP OF TWO SPACES WITH A COMMON POINT: A CORRECTION

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IN my paper 'The fundamental group of two spaces with a common point' [this Journal, 5 (1954), 175–90] the proof of Lemma 3.1 contains a fault on p. 184, line 21, where the use of Zoretti's theorem presupposes that the set F there is connected. The correct procedure is as follows (with the same notation as before).

The vertex v of the cone $\hat{A}$ has $\hat{A}-A$ as open neighbourhood. Hence there exists $\delta > 0$ such that $\psi U(F,\delta) \subseteq \hat{A}-A$. Let the sets

$$V_i = U(x_i, \delta/2), \qquad x_i \in F \quad (1 \leqslant i \leqslant n)$$

be a finite covering of the compact set F and let the components of $\bar{V}_1 \cup \bar{V}_2 \cup \ldots \cup \bar{V}_n$ be $W_1, W_2, \ldots, W_r$, where $r \leqslant n$ and each W_i is the union of certain of the $\bar{V}_j$. Let $d > 0$ be the minimum distance between the W_i, and let $\eta = \min(\frac{1}{3}d, \frac{1}{4}\delta)$. Applying Zoretti's theorem to each connected set W_i, let J_i be a Jordan curve in $U(W_i, \eta)$ such that $W_i \subseteq \operatorname{int} J_i = I_i$. Then $J_i \subseteq U(F,\delta)$, so that $\psi J_i \subseteq \hat{A}-A-v$. Orient J_i like $\dot{E}^2$.

In E^2, we had previously chosen $c \in \dot{E}^2$ with $\psi(c) = x \in A$ as base point of homotopy groups. Let μ_i be an arc joining c to $d_i \in J$, and lying in $E^2 - \bigcup I_i$ $(1 \leqslant i \leqslant r)$, and let $\theta\colon \dot{E}^2 \to E^2$ be the identity mapping. Then, by induction on r, we obtain

$$\theta \simeq (\mu_1 J_1 \mu_1^{-1})(\mu_2 J_2 \mu_2^{-1})\ldots(\mu_r J_r \mu_r^{-1}) \operatorname{rel} c \text{ in } E^2 - \bigcup I_i,$$

so that

$$f = \psi\theta \simeq (\lambda_1 L_1 \lambda_1^{-1})(\lambda_2 L_2 \lambda_2^{-1})\ldots(\lambda_r L_r \lambda_r^{-1}) \operatorname{rel} x \text{ in } Y - v,$$

where

$$\lambda_i = \psi(\mu_i), \qquad L_i = \psi(J_i) \quad (1 \leqslant i \leqslant r).$$

We now apply the mapping $\phi\colon Y - v \to X$, to get

$$f = \phi f = \phi\psi f \simeq (\nu_1 M_1 \nu_1^{-1})\ldots(\nu_r M_r \nu_r^{-1}) \operatorname{rel} x \text{ in } X,$$

where

$$\phi(\lambda_i) = \nu_i, \qquad \phi(L_i) = M_i \quad (1 \leqslant i \leqslant r).$$

Since A is path-wise connected, there is a path $\kappa_i \subseteq A$ from $\phi\psi(d_i)$ to x; whence, in X,

$$(a) \quad (\nu_1 M_1 \nu_1^{-1})\dots(\nu_r M_r \nu_r^{-1}) \simeq (\nu_1 \kappa_1)\kappa_1^{-1} M_1 \kappa_1 (\nu_1 \kappa_1)^{-1} \dots$$
$$\dots(\nu_r \kappa_r)\kappa_r^{-1} M_r \kappa_r (\nu_r \kappa_r)^{-1} \operatorname{rel} x.$$

But $(\nu_i \kappa_i)$ is a loop in X, through x; while $\kappa_i M_i \kappa_i^{-1}$ is a loop through x, and in A since $\kappa_i \subseteq A$ and $M_i = \phi\psi(J_i) \subseteq \phi(\hat{A} - v) = A$. Therefore the homotopy class of f in $\pi_1(X)$ contains the right-hand member of (a), and so is in $\{j_1 \pi_1(A)\}^N$. The proof now proceeds as in the original.

The sentence forming footnote ‡ on p. 187 should be added to footnote § on the same page; footnote ‡ should be replaced by 'See footnote ‡ on p. 186'.

SOME RAPIDLY CONVERGENT SERIES FOR THE RIEMANN ξ-FUNCTION

By A. P. GUINAND (*Shrivenham*)

[Received 9 March 1955]

1. Introduction

It is known that there exists a summation formula†

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)f(n)-\zeta(1-s)\int_0^{\infty} x^{-\frac{1}{2}s}f(x)\,dx-\zeta(1+s)\int_0^{\infty} x^{\frac{1}{2}s}f(x)\,dx$$

$$=\sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)g(n)-\zeta(1-s)\int_0^{\infty} x^{-\frac{1}{2}s}g(x)\,dx-\zeta(1+s)\int_0^{\infty} x^{\frac{1}{2}s}g(x)\,dx, \quad (1)$$

where $\sigma_s(n)$ is the sum of the sth powers of the divisors of n, $f(x)$ satisfies appropriate conditions, and $g(x)$ is the transform of $f(x)$ with respect to the Fourier kernel

$$-2\pi\sin\tfrac{1}{2}s\pi\, J_s(4\pi x^{\frac{1}{2}})-\cos\tfrac{1}{2}s\pi\{2\pi Y_s(4\pi x^{\frac{1}{2}})-4K_s(4\pi x^{\frac{1}{2}})\}.$$

If we choose $f(x)$ so that both $f(x)$ and $g(x)$ decrease rapidly as x increases, then (1) gives a formula connecting $\zeta(1+s)$ and $\zeta(1-s)$ with rapidly convergent series. In this paper I give several such results, which can then be combined to give a formula for the Riemann ξ-function,

$$\xi(s)=\tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s).$$

The formula is

$$\xi(s)=4\pi(1-s)\sum_{n=1}^{\infty} n^{1-\frac{1}{2}s}\sigma_s(n)K_{1+\frac{1}{2}s}(2n\pi)+$$

$$+16\pi^3 s^{-1}\sum_{n=1}^{\infty} n^{3-\frac{1}{2}s}\sigma_s(n)\{(s-7)K_{1+\frac{1}{2}s}(2n\pi)+(s+7)K_{1-\frac{1}{2}s}(2n\pi)\}, \quad (2)$$

and is valid for $s\neq 0$.

Now
$$n^{-\frac{1}{2}s}\sigma_s(n)=O(n^{\frac{1}{2}|\sigma|+\epsilon}),$$
where $\sigma=\operatorname{re}s$, and, for fixed μ,
$$K_\mu(2n\pi)\sim\tfrac{1}{2}n^{-\frac{1}{2}}e^{-2n\pi}$$
as $n\to\infty$. Also $e^{2\pi}\simeq 535{\cdot}5$.

Hence the series (2) converges at least as rapidly as the series

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}(|\sigma|+5)+\epsilon}(535{\cdot}5)^{-n}.$$

† A. P. Guinand, *Quart. J. of Math.* (Oxford) 10 (1939), 104–18, Theorem 6.

Quart. J. Math. Oxford (2), 6 (1955), 156–60.

As the summation formula (1) is not very well known, independent proofs of the results are given. Some simpler formulae corresponding to $s = 3, 5, 7, \ldots$ are given in § 4.

2. The basic formula

The formula from which the result (2) is derived is given by the theorem:

THEOREM. *For* $\operatorname{re} z > 0$ *and for all* s *except* 0 *or* ± 1,

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)K_{\frac{1}{2}s}(2\pi nz) - z^{-1}\sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)K_{\frac{1}{2}s}(2\pi n/z)$$
$$= \frac{\xi(s)}{2s(s-1)}(z^{\frac{1}{2}s-1}-z^{-\frac{1}{2}s}) + \frac{\xi(-s)}{2s(s+1)}(z^{-\frac{1}{2}s-1}-z^{\frac{1}{2}s}). \quad (3)$$

To prove this result, consider the formula†

$$\tfrac{1}{4}\Gamma(\tfrac{1}{2}s)(\pi y)^{-\frac{1}{2}s} + \sum_{n=1}^{\infty} n^{\frac{1}{2}s}K_{\frac{1}{2}s}(2\pi ny)$$
$$= \tfrac{1}{4}\pi^{-\frac{1}{2}s-\frac{1}{2}}y^{-\frac{1}{2}s-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s) + \tfrac{1}{2}\pi^{-\frac{1}{2}s-\frac{1}{2}}y^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\sum_{n=1}^{\infty}(y^2+n^2)^{-\frac{1}{2}-\frac{1}{2}s},$$

valid for $\operatorname{re} s > 0$, $\operatorname{re} y > 0$.

If we put $y = mz$, multiply throughout by $m^{-\frac{1}{2}s}$, and sum over positive integral m, then, for $\operatorname{re} s > 1$,

$$\tfrac{1}{4}\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}z^{-\frac{1}{2}s}\sum_{m=1}^{\infty} m^{-s} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-\frac{1}{2}s}n^{\frac{1}{2}s}K_{\frac{1}{2}s}(2\pi mnz)$$
$$= \tfrac{1}{4}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\pi^{-\frac{1}{2}s-\frac{1}{2}}z^{-\frac{1}{2}s-1}\sum_{m=1}^{\infty} m^{-s-1} +$$
$$+\tfrac{1}{2}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\pi^{-\frac{1}{2}s-\frac{1}{2}}z^{\frac{1}{2}s}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2z^2+n^2)^{-\frac{1}{2}-\frac{1}{2}s}.$$

That is,

$$\tfrac{1}{4}\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s) - \tfrac{1}{4}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\pi^{-\frac{1}{2}s-\frac{1}{2}}z^{-1-\frac{1}{2}s}\zeta(1+s) + \sum_{n=1}^{\infty} n^{-\frac{1}{2}s}\sigma_s(n)K_{\frac{1}{2}s}(2n\pi z)$$
$$= \tfrac{1}{2}\pi^{-\frac{1}{2}s-\frac{1}{2}}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)S(z), \quad (4)$$

where
$$S(z) = z^{\frac{1}{2}s}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2z^2+n^2)^{-\frac{1}{2}-\frac{1}{2}s},$$

and all the series concerned converge absolutely for $\operatorname{re} s > 1$, $\operatorname{re} z > 0$.

† This can be derived from Poisson's summation formula. Cf. G. N. Watson, *Quart. J. of Math.* (Oxford) 2 (1931), 298–309.

Further,
$$\frac{1}{z}S\left(\frac{1}{z}\right) = z^{-\frac{1}{2}s-1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2z^{-2}+n^2)^{-\frac{1}{2}-\frac{1}{2}s}$$
$$= z^{\frac{1}{2}s}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2z^2)^{-\frac{1}{2}-\frac{1}{2}s}$$
$$= S(z). \tag{5}$$

Substituting the left-hand side of (4) in (5), we obtain (3) for $\operatorname{re} s > 1$, $\operatorname{re} z > 0$. Since all the series in (3) are uniformly convergent series of analytic functions† of s in any finite region of values of s, it follows by analytic continuation that the result is true for all s except $s = 0, \pm 1$.

If we make s approach 0, we obtain Koshliakov's formula,‡
$$\sum_{n=1}^{\infty} d(n)K_0(2\pi nz) - z^{-1}\sum_{n=1}^{\infty} d(n)K_0(2\pi n/z)$$
$$= \tfrac{1}{4}\{\log(4\pi/z)-\gamma\} - \tfrac{1}{4}z^{-1}\{\log(4\pi z)-\gamma\}.$$

If we make s approach ± 1, we obtain the formula§
$$\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi nz} - \sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n/z} = \tfrac{1}{12}\pi(z^{-1}-z)+\tfrac{1}{2}\log z. \tag{6}$$

3. Formulae for $\xi(s)$

Formulae for $\xi(s)$ can obviously be derived from (3) by taking two different values of z in (3) and eliminating the terms in $\xi(-s)$ from the two formulae. However, more symmetrical results with more rapidly convergent series are obtained by differentiating (3) an odd number‖ of times with respect to z and then putting $z = 1$. The differentiation of the series is justified by the absolute and uniform convergence of the differentiated series.

Differentiating (3) once and putting $z = 1$, we find that
$$\tfrac{1}{2}\xi(s) - \tfrac{1}{2}\xi(-s) = 2\pi(1-s)\sum_{n=1}^{\infty} n^{1-\frac{1}{2}s}\sigma_s(n)K_{1+\frac{1}{2}s}(2n\pi) -$$
$$-2\pi(1+s)\sum_{n=1}^{\infty} n^{1-\frac{1}{2}s}\sigma_s(n)K_{1-\frac{1}{2}s}(2n\pi). \tag{7}$$

† E. C. Titchmarsh, *Theory of Functions* (Oxford, 1939), 95.

‡ N. S. Koshliakov, *Messenger of Math.* (2) 58 (1928), 30–32.

§ This is equivalent to the well-known result
$$\prod_{n=1}^{\infty}(1-e^{-2\pi nz}) = z^{-\frac{1}{2}}\exp\tfrac{1}{12}\pi(z-z^{-1})\prod_{n=1}^{\infty}(1-e^{-2\pi n/z}).$$

‖ If we differentiate an even number of times and put $z = 1$, then both sides vanish.

Differentiating (3) thrice and putting $z = 1$, we find that

$$\tfrac{1}{2}\xi(s)+\tfrac{1}{2}\xi(-s) = 2\pi(1-s)\sum_{n=1}^{\infty} n^{1-\frac{1}{2}s}\sigma_s(n)K_{1+\frac{1}{2}s}(2n\pi)+$$

$$+2\pi(1+s)\sum_{n=1}^{\infty} n^{1-\frac{1}{2}s}\sigma_s(n)K_{1-\frac{1}{2}s}(2n\pi)+$$

$$+16\pi^3 s^{-1}(s-7)\sum_{n=1}^{\infty} n^{3-\frac{1}{2}s}\sigma_s(n)K_{1+\frac{1}{2}s}(2n\pi)+$$

$$+16\pi^3 s^{-1}(s+7)\sum_{n=1}^{\infty} n^{3-\frac{1}{2}s}\sigma_s(n)K_{1-\frac{1}{2}s}(2n\pi). \tag{8}$$

Finally, adding (7) and (8), we obtain the formula (2).

4. Series for $\zeta(2n+1)$

In an earlier paper† I noted that it is possible to derive formulae similar to (6) which involve values of $\zeta(s)$ for $s = 3, 5, 7,\ldots$.

These formulae can be written

$$\sum_{n=1}^{\infty}\sigma_{-2p+1}(n)e^{-2\pi nz}+(-1)^p z^{2p-2}\sum_{n=1}^{\infty}\sigma_{-2p+1}(n)e^{-2\pi n/z}$$

$$= \frac{(-1)^{p+1}(2\pi)^{2p-1}}{2z(2p)!}(B'+izB)^{2p}-\tfrac{1}{2}\zeta(2p-1)\{1+(-1)^p z^{2p-1}\}, \tag{9}$$

where $\operatorname{re} z > 0$, p is an integer greater than 1, and terms of the form $(B')^m(B)^n$ are to be interpreted‡ as $B_m\,B_n$.

Now
$$\sum_{n=1}^{\infty}\sigma_{-2p+1}(n)e^{-2\pi nz} = \sum_{n=1}^{\infty}\frac{1}{n^{2p-1}(e^{2\pi nz}-1)}.$$

Hence, if p is even, $p = 2k$ say, and we put $z = 1$ in (9), we have§

$$\zeta(4k-1) = -\frac{(2\pi)^{4k-1}}{2(4k)!}(B'+iB)^{4k}-2\sum_{n=1}^{\infty}\frac{1}{n^{4k-1}(e^{2n\pi}-1)} \tag{10}$$

for positive integral k.

† A. P. Guinand, *Quart. J. of Math.* (Oxford) 15 (1944), 11–23, Theorem 9 (iv).

‡ The use of this notation in a similar case is due to H. F. Sandham, *Proc. American Math. Soc.* 5 (1954), 430–6. B_n is defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n.$$

§ An equivalent formula is given by H. F. Sandham, loc. cit. 6.32.

If p is odd, $p = 2k+1$ say, this method fails. If we first differentiate (9) with respect to z and then put $z = 1$, we find that

$$\zeta(4k+1) = -\frac{(2\pi)^{4k+1}}{4k(4k+1)!}\, iB(B'+iB)^{4k+1} - $$
$$-2\sum_{n=1}^{\infty}\frac{1}{n^{4k+1}(e^{2n\pi}-1)} - \frac{2\pi}{k}\sum_{n=1}^{\infty}\frac{e^{2n\pi}}{n^{4k}(e^{2n\pi}-1)^2} \qquad (11)$$

for positive integral k.

A few terms of the series (10) and (11) suffice to compute numerical values of $\zeta(s)$ for positive odd s if $e^{2\pi}$ and π are known to sufficient accuracy.†

5. Allied results

A formula which can also be regarded as a special case of (1) is discussed by Atkinson,‡ but the series involved are not convergent.

Formulae connected with $\xi(s)$ which involve Bessel functions in much the same way as (2) have been discussed by several writers.§

A generalization of (9) in which the term corresponding to $\zeta(2p-1)$ does not occur is given by Apostol.‖

† $e^{2\pi} = 535{\cdot}54916\ 55524\ 76473\ 65030\ 49326$ to twenty-five places of decimals.

‡ F. V. Atkinson, *Duke Math. J.* 17 (1950), 63–68.

§ Cf. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function* (Oxford, 1951), 213–14 and 237–40.

‖ T. M. Apostol, *Duke Math. J.* 17 (1950), 147–57, Theorem 2.

THE ANALYSIS OF OBSERVATIONS (II)

By C. W. KILMISTER (*London*)

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1. Introduction

In a recent paper (**1**) a new approach to Eddington's wave-tensor calculus was suggested. The basis of it was the concept of scale invariance. It was found that, when certain simplifying assumptions [(**1**) § 5] are made, corresponding to those made in quantum theory, familiar results appear. In the present paper these simplifying assumptions are not made. One of the applications of the more general theory which results is to the proof of Eddington's result [(**3**) (15.51)] on the division of a property between particles and field. Eddington's discussion of this is not satisfactory. The result is needed in the deduction of his equation [(**3**) (16.5)]

$$\frac{m_1}{m_2} = \frac{k_2}{k_1}.$$

This depends also on other matters to be investigated.

Since the appearance of (**1**) a more fundamental investigation has been made (**2**). It should be made clear that the present paper is strictly a continuation of the ideas of (**1**), and that its only connexion with the later work is indirect.

It is found that the ideas involved are logically complex. As a result, some formalization of the argument is essential. This is also an advantage in ensuring that no implicit assumptions enter the theory.

2. Discreteness

The subject-matter of the present paper may be differentiated roughly from that of (**1**) by remarking that the observables considered in (**1**) are constant, but here I intend to consider variables. Before we can do this we have therefore to deal with a problem which arises in any theory of measurement, but of which no discussion suitable for our needs has been given. Consider, for example, a real variable x representing part of the result of an observation. Since physical results are expressed to a finite number of decimal places, x is a discrete variable. We wish to try to distinguish two kinds of discreteness. If x is expressed to n decimal places, let $X(n)$ be the number of values of x in a range (a, b). In

mathematics we should say that 'x is discrete in (a, b)' if $X(n)$ is bounded above as $n \to \infty$.

If $X(n)$ is not bounded, then x is continuous at one point at least of (a, b). We notice, however, that this gives no means of determining experimentally whether x is discrete or continuous. The usual assumption in physics seems to be that all variables are continuous until evidence to the contrary is produced. Accordingly we make the definition

[2.1] Definition. *If x is a real variable, expressed to n decimal places, $X(n)$ the number of values of x in a range (a, b), and if we know that $X(n)$ is bounded above, x is called 'discrete' in (a, b).*

(Note: I shall number definitions, axioms, and theorems consecutively.)

[2.2] Definition. *With the notation of [2.1], if we know of no upper bound to $X(n)$, x is called 'pseudo-continuous' in (a, b).*

The property of pseudo-continuity is a property of the whole interval (a, b). Hence we make the definition:

[2.3] Definition. *If x is pseudo-continuous in every sub-interval of (a, b), it is called continuous in (a, b).*

The following theorems follow at once:

[2.4] Theorem. *Discreteness in the sense of [2.1] implies analytical discreteness.*

[2.5] Theorem. *Analytical continuity implies continuity in the sense of [2.3].*

In virtue of these two results it is usual to approximate to a continuous variable by one which is continuous in the analytic sense.

3. Notation: measurement conditions

As we need to formalize the theory somewhat in order to make it clear, the following notations, familiar in algebra, will be used:

(x)	for all x
$(\exists x)$	there exists an x
$\Rightarrow$	implies
$x \in A$	x belongs to A
$A = \mathfrak{S}(x \mid P)$	A is the set of all x satisfying P.

In (**1**) 559–66 a discussion of measurement conditions was given. Reference should be made to this for a discussion of the physical basis of this portion of the theory. However, for the purposes of this paper the following weaker form of the definitions and results of (**1**) is sufficient:

[3.1] DEFINITION. *An observable is the result of a physical observation.*

[3.2] AXIOM. *An observable of order n can be represented by an element of an algebra $\mathscr{C}$ of order n over the real field. This algebra has a basis*

$$\{e_1, e_2, \ldots, e_{n-1}; e_n = 1\}$$

such that $e_1^2 = 1$. The automorphisms leaving e_1 invariant represent changes of the reference system.

[3.3] DEFINITION. *The scale σ of an observable ϕ is defined by*

$$\phi = \sum_{r=1}^{n} \phi_r e_r, \qquad \sigma(\phi) = \phi_1.$$

4. Variables

In this paragraph we consider a further condition imposed on observables. It derives from the paradox that 'the system measured must be at the same time an isolated whole and a part interacting with other parts' [Temple (**4**)]. Since the operation of measurement may change the observable, we cannot expect that two successive determinations of an observable ϕ will give the same result. The axiom [3.2] will still be true in the sense that the result of any one observation is an element of $\mathscr{C}$. In (**1**) a special device (that of 'simultaneous particles') was employed to simplify the analysis. This device is equivalent to the usual assumptions of quantum theory and so leads to results of a familiar form. In the present paper I shall not restrict the analysis in this way.

We consider the set ϕ^* (say) of all observations of ϕ. Now ϕ^* is an entity which transcends our actual observation. By its introduction the theory is formally extended. It will appear from the results that such an extension is implicit in Eddington's work. It seems that his failure to state it explicitly is responsible for some of the difficulty of 'Fundamental Theory'. We define, taking into account scale-conditions as in (**1**) 567:

[4.1] DEFINITION. $\phi^* = \mathfrak{S}\{\phi^i \mid \sigma(\phi^i) = \lambda\}$.

Here λ is the common scale of all the observables ϕ^i. It is thus convenient to write

[4.2] DEFINITION. $\sigma(\phi^*) = \sigma(\phi^i) = \lambda$.

In these definitions the ϕ^i belong to some set of observations. In general it is false that ϕ^* is observable, in the sense of [3.1], [3.2]. (It is true only if ϕ^* is a finite set.) On the other hand ϕ^* does correspond to our intuitive idea of an observable which can have a number of values. Accordingly

we define entities like ϕ^* to be observables of the first kind. If we introduce the notations

[4.3] DEFINITION. $O(\phi) \equiv [\phi$ *is observable*] (*in the sense of* [3.1], [3.2]),

[4.4] DEFINITION. $O'(\phi^*) \equiv [\phi^*$ *is an observable of the first kind*],

these two will be related by

[4.5] THEOREM. $O'(\phi^*) \equiv [\phi^* = \mathfrak{S}\{\phi^i \mid \sigma(\phi^i) = \lambda;\ O(\phi^i)\}]$.

Observables of the first kind correspond to variables in the analysis; but they can obviously have only an enumerable number of values, for a set of experiments can be infinite only by virtue of future repetition. We can introduce the idea of discreteness for these observables.

[4.6] DEFINITION. *Let* $\tau \subseteq \phi^*$; *if*

$$\phi^i \in \tau \Rightarrow (\mu)\ [\phi^i_\mu \textit{ is discrete}],$$

we call ϕ^* '*discrete in* τ'. *But, if*

$$\phi^i \in \tau \Rightarrow (\exists\, \mu)\ [\phi^i_\mu \textit{ is continuous}],$$

we call ϕ^* '*continuous in* τ'.

Thus 'discrete' and 'continuous', applied to observables, are mutually exclusive properties. In particular

[4.7] DEFINITION. *If* $\tau = \phi^*$, *we say simply that* ϕ^* *is 'discrete', or 'continuous'*.

5. Observables of the second kind

From [3.2], [3.3], [4.5] we have

$$O'(\phi^*) \equiv \left[\phi^* = \mathfrak{S}\left\{ \sum_{\mu=1}^{n} \phi^i_\mu e_\mu \,\middle|\, \phi^i_1 = \lambda;\ O(\phi^i)\right\}\right].$$

We now define a set of such observables of the first kind.

[5.1] DEFINITION. $\psi^*_x = \mathfrak{S}\{\phi^i + xe_1 \mid \phi^i_1 = \lambda;\ O(\phi^i)\}$,

[5.2] DEFINITION. $\Psi = \mathfrak{S}\{\psi^*_x \mid O'(\psi^*_x)\}$.

We shall call Ψ an 'observable of the second kind'. For, although it is not an observable either in the sense of [3.1], [3.2] or in that of [4.5], it corresponds to our idea of an observable system which may have more than one value of the scale. In the same way as in [4.6] we define

[5.3] DEFINITION. *Let* $T \subseteq \Psi$; *if*

$$\psi^*_x \in T \Rightarrow \psi^*_x \textit{ is discrete},$$

we call Ψ '*discrete in* T'. *But, if*

$$\psi^*_x \in T \Rightarrow \psi^*_x \textit{ is continuous},$$

we call Ψ '*continuous in* T'.

In particular

[5.4] DEFINITION. *If $T = \Psi$, we say simply that Ψ is 'discrete' or 'continuous'.*

Eddington has considered quantities which he calls 'scale-fixed' and 'scale-free' [(**3**) 17]. Of the quantities defined in this paper, these properties could be predicted only of observables of the second kind. We conclude that Eddington is, in fact, dealing with such observables. Following Eddington we define:

[5.5] DEFINITION. $\Psi = \mathfrak{S}\{\psi_x^* \mid O'(\psi_x^*)\}$

is scale-free if

$$O'(\psi_{x_0}^*) \Rightarrow (\exists\, \delta)[|x-x_0| < \delta \Rightarrow O'(\psi_x^*)].$$

$$\Psi = \mathfrak{S}\{\psi_x^* \mid O'(\psi_x^*)\}$$

is scale-fixed if x is discrete.

We notice that one of these definitions is given in terms of analytic continuity. This is necessary in what follows. The loss in physical connexion is small since observables of the second kind are not directly observable. We shall not be able to determine experimentally whether a given observable is or is not scale-free. But, since the definition by physical continuity would have settled this by assuming an observable scale-free unless contrary evidence was known, it really provides no more.

Eddington [(**3**) §§ 14, 15] divides observables into discrete ones, which are by implication scale-fixed, and scale-free ones, which are continuous. The following results justify this division:

[5.6] THEOREM. *If Ψ is scale-free, it is continuous (in itself).*

Proof. To prove that Ψ is continuous we have, by [5.3], to prove ψ_x^* continuous: that is, by [5.1], that

$$\mathfrak{S}\{\phi^i + xe_1 \mid \sigma(\phi^i) = \lambda;\, O(\phi^i)\}$$

is continuous. Thus, by [4.6], we have to prove that

$$(\exists\, \mu)\ [\phi_\mu^i + x\,\delta_{\mu 1} \text{ is continuous}].$$

Such a value is $\mu = 1$, for, in fact, $\phi_1^i + x$ is analytically continuous by [5.5] since Ψ is scale-free, and the result follows by [2.5].

[5.7] THEOREM. *If Ψ is discrete (in itself), it is scale-fixed.*

Proof. By [5.3], ψ_x^* is discrete. Hence, by [4.6],

$$(\mu)\ [\phi_\mu^i + \delta_{\mu 1} x \text{ is discrete}].$$

Hence x is discrete, which proves the result, by [5.5].

Eddington's division between discrete and scale-free observables leaves out of account the class of scale-fixed continuous observables. These are mathematically possible. However, they correspond to a physically uncommon state of affairs. For, if we take x as fixed (for example $x = x_0$), we must have $\psi^*_{x_0}$ continuous. Thus $\mathfrak{S}(\psi^i_{x_0})$ is continuous: that is, we have a sequence whose points are continuous. Such sequences do not seem to occur in physics.

6. Phase-space

Although observables of the second kind are physically quite different from those of the first, the mathematical difference is slight. Those of the first kind correspond to a set of simultaneous observables; those of the second kind to a set without the restriction of simultaneity. This difference is clear in a geometrical picture. Let E_n be a space of n dimensions. Any simple observable ϕ, of order n, can be represented by a point (ϕ_μ) in E_n referred (say) to a Cartesian coordinate system. An observable of the first kind ϕ^* is then represented by a set of points all lying in an $(n-1)$-flat $\phi_1 =$ constant. The set Ψ of observables is represented by a set of points in a family of parallel $(n-1)$-flats. If Ψ is scale-fixed, these $(n-1)$-flats are discrete. If Ψ is scale-free, they may fill a volume.

In (**1**) 'rotations' in the set of observables were defined [(**1**) 560] as transformations corresponding only to changes in the reference system. Such rotations induce transformations of the space E_n into itself. Further, these transformations leave the scale invariant [(**1**) 561]; accordingly a different phase-space is more convenient. If E_{n-1} is a space of $n-1$ dimensions, an observable ϕ of order n can be represented by the point

$$\xi_r = \frac{\phi_r}{\phi_1} \quad (r = 2, 3, \ldots, n),$$

except for its scale. An observable of the first kind ϕ^* is then represented by a general set of points in E_{n-1}. The sets representing ψ^*_x are derived from that representing ϕ^* by expansion about the origin.

We have thus a set of points in E_{n-1} representing any observable of the first kind. It is now natural to introduce a probability view. This may be done in the usual way; only some care is necessary in order to avoid overstepping the limits of experimental knowledge. Let V be a given volume in E_{n-1}, let the number of points in V after N measurements be $n(V)$, and define

$$P(V, N) = n(V)/VN.$$

Then

$$P(V, N+1) = (n+\alpha)/V(N+1),$$

where $\alpha = 0$ or 1. Hence

$$P(V,N+1)-P(V,N) = (\alpha N-n)/VN(N+1).$$

Thus $$|P(V,N+1)-P(V,N)| \leqslant 1/V(N+1).$$

Hence, for fixed V, the sequence $P(V,N)$ is convergent, to a limit $P(V)$, say. It should be noted that the convergence is proved in the usual analytical sense, although P is the result of observation. We can now find a sequence $P(V)$, $P(V_1)$,..., $P(V_s)$, where (V_s) is a sequence of volumes with a common point (ξ_μ), and

$$V_1 = \frac{V}{2^{n-1}}, \qquad V_{s+1} = \frac{V_s}{2^{n-1}}.$$

But we have in this case no guarantee of convergence. If the sequence is convergent, we can write for its limit $p(\xi_\mu)$. Thus we make the definition

[6.1] Definition. *A point at which $P(V_s)$ converges is a regular point.*

[6.2] Theorem. (ξ_μ *is regular*) $\equiv [\lim P(V_s) = p(\xi_\mu)]$.

[6.3] Theorem. *The probability that a measurement of ϕ gives a result in $d\xi_2 d\xi_3 ... d\xi_n$ is*

$$p(\xi_\mu)\, d\xi_2\, d\xi_3 ... d\xi_n.$$

The position is now this: we have an observable ϕ. Any measurement of ϕ yields a point in the set ϕ^*, but we have no information as to which point. Any value of the observable is particular information, in the sense that it is knowledge of the result of a past experiment, but is not useful in predicting the result of a future one. None the less, we can define the probability function p. This gives general information about the observable since it is useful in predicting the result of a future experiment.

Consider now two observables, both of order n, whose probability functions are equal everywhere. If we perform single measurements of both these observables, the results will not in general be equal. But, if we perform a number of measurements of each, the two sets of results will have the same probability distributions. Thus the two observables cannot be said to be physically unequal. It follows that the probability distribution gives all the general information possible about an observable. Explicitly:

[6.4] Definition. *Two observables of the first kind ϕ^*, ϕ'^* are equivalent,*

$$\phi^* \sim \phi'^*,$$

if they have the same probability distribution.

It is obvious that [6.4] defines an equivalence relation.

[6.5] Axiom. *Equivalent observables with the same scales are physically indistinguishable.*

We can now define a special class of observables of the second kind. Starting with ψ_x^* defined as in [5.1], we define

[6.6] Definition. $\Psi^0 = \mathfrak{S}\{\psi_x^* \mid O'(\psi_x^*); \psi_x^* \sim \phi^*\}$.

Thus, by [6.5], Ψ^0 represents a set of observables which are physically distinguishable only by their having different scales.

[6.7] Definition. *With the notation of* [6.6], Ψ^0 *is a reducible observable of the second kind.*

There is no doubt from the context that Eddington is considering only reducible observables when he uses scale-free as contrasted with scale-fixed properties. In § 5 we had not formulated enough properties to be able to define his concept completely. The form of [5.5] corresponds exactly to his explicit statement [(**3**) 17], when taken with reducibility.

The definitions above have been made assuming that possible values of ϕ^* form an $(n-1)$-dimensional set. It may happen that they form only a k-dimensional set, where $k < n-1$. The definitions of probability must then be modified in the usual way, by taking a k-dimensional volume-element for V.

Let Ψ be a scale-free observable of the second kind. Let $\phi^* \in \Psi$ define the phase-space E_{n-1} and probability distribution p of ϕ^* as above. Consider now the observable of the first kind ψ_x^* $(x \neq 0)$. For a given x, there is a value ψ_x corresponding to each value ϕ. The result of the scale-change, however, is to cause an expansion of E_{n-1} about the origin, so that the volume element $d\tau$ for ϕ becomes $\lambda^{n-1}\,d\tau$ for ψ_x, where

$$\sigma(\psi_x) = \sigma(\phi)/\lambda,$$

if the possible values of ψ_x^* form an $(n-1)$-dimensional set. If they form only a k-dimensional set, the new volume element will be $\lambda^k\,d\tau$. Since values of ψ_x^*, ϕ^* are in (1:1) correspondence, the probability of finding a value ϕ in a volume $d\tau$ is the same as that of finding a value ψ_x in a volume $d\tau' = \lambda^k\,d\tau$.

Hence $$p' = \lambda^{-k}p.$$

But $$p' = p'(\xi'_\mu) = p'(\lambda\xi_\mu).$$

Thus we have established, for scale-free systems,

$$p'(\lambda\xi_\mu) = \lambda^{-k}p(\xi_\mu).$$

We deduce

[6.8] THEOREM. *If Ψ is scale-free and reducible, p is a homogeneous function of the ξ_μ of degree $-k$, where k is the number of dimensions of the set representing Ψ.*

For, if Ψ is reducible, then

$$\psi_x^* \sim \phi^*,$$

and so
$$p'(\lambda\xi_\mu) = p(\lambda\xi_\mu) = \lambda^{-k}p(\xi_\mu).$$

7. Functions of observables

We wish now to define the concept of a scalar function of an observable in such a way as to be physically significant. If this is to be so, the function must have the same value for two values of the argument which are physically indistinguishable. Thus, from [6.5] we have

[7.1] DEFINITION. $[\phi^* \sim \phi'^*;\ \sigma(\phi^*) = \sigma(\phi'^*)] \Rightarrow F(\phi^*) = F(\phi'^*)$, *where F is such a function of observables.*

It follows from [7.1], with [6.4], that $F(\phi^*)$ depends only on the scale and probability distribution of ϕ^*. It is convenient to limit the definition [7.1] somewhat. With the notation of [5.1], [5.2], we define:

[7.2] DEFINITION. *F is separable if*

$$F(\psi_x^*) = g(x)H(\phi^*),$$

where $g(x)$ is an ordinary function of x.

[7.3] THEOREM. *If F is separable, we can write*

$$F(\psi_x^*) = G(x)F(\phi^*), \qquad G(0) = 1.$$

For, in fact, from [7.2], putting $x = 0$, we have

$$F(\phi^*) = g(0)H(\phi^*),$$

and then [7.3] follows with $G(x) = g(x)/g(0)$. It follows at once, by Taylor's theorem, that

[7.4] THEOREM. *If (a) F is separable,*

(b) Ψ is scale-free and reducible,

(c) G is continuous and differentiable twice,

then
$$F(\psi_x^*) = \{1+c_1 x+O(x^2)\}F(\phi^*),$$

where
$$c_1 = G'(0).$$

However, we still have a very large class of functions. In order to deduce further results, some particular cases must be chosen. One such case is provided by

[7.5] Definition. *$F(\phi^*) = \int\limits_{E_k} f(p, p_\mu, p_{\mu\nu},...)\, d\tau$, where $d\tau$ is a volume-element of the dimensions of the set E_k representing ϕ^*, p is the probability distribution of ϕ^*, and*

$$p_\mu = \frac{\partial p}{\partial \xi_\mu}, \qquad p_{\mu\nu} = \frac{\partial^2 p}{\partial \xi_\mu \partial \xi_\nu}, \text{ etc.}$$

It is natural to start by considering such functions for the physical reason that, if f depends on all the differential coefficients of p of arbitrarily high order, then its value at ξ_μ can be regarded, in a sense, as depending on the value of p at all points.

If F is separable, [7.4] must be satisfied.

Now we have

[7.6] Theorem. *If $[\phi^* \to \psi_x^*] \equiv [f \to \{1+c_2 x+O(x^2)\}f]$, then [7.4] is satisfied.*

For then
$$d\tau \to d\tau' = \lambda^k\, d\tau = \{1+c_3 x+O(x^2)\}\, d\tau,$$
which proves the result.

The restriction of [7.6] is the most convenient way of satisfying [7.4], and we accordingly define

[7.7] Definition. *If $[\phi^* \to \psi_x^*] \equiv [f \to \{1+c_2 x+O(x^2)\}f]$, f is said to 'generate, by [7.5], a Hamiltonian property F'.*

Consider two observables with Hamiltonian properties in which $c_1 = c_1'$, so that the properties of the whole sets of points change in a similar manner for a small scale-change. We make the definition

[7.8] Definition. *If $c_1 = c_1'$, F, F' are extrinsically equivalent.*

This is obviously an equivalence relation. If, however, $c_2 = c_2'$, so that f, f' change similarly, then properties of each point of the sets change in the same way. We make the definition

[7.9] Definition. *If $c_2 = c_2'$, F, F' are intrinsically equivalent.*

Since $d\tau$ is a homogeneous function of the scale of degree k, we have

[7.10] Theorem. *If and only if the two observables have the same dimensionality, intrinsic equivalence implies extrinsic.*

The above definitions deal with scalar functions of observables. However, exactly similar results apply to finite sets of such scalar

functions. It follows that if we are given some 4-dimensional space-time, we can define tensor functions of observables in this space-time. (There is no question here of transformation properties.)

8. Carriers: fields

If we are given a set of variables (e.g. an observable) which occur together, it is an assistance in considering the theory to postulate a carrier of the set. This is harmless so long as we introduce no other properties of the carrier. For the carrier of a set is only a convenient name for the set itself. Any carrier is called a 'particle' by Eddington [(**3**) 30]. It should be noticed that these definitions of 'carrier' and 'particle' are more general than those given in (**1**) [568, 570]. The restricted definitions are appropriate to the special theory given there. We have to consider the more general ones. This use of the word 'particle' is justified since it is equivalent, as near as can be judged, to the use in contemporary quantum theory.

We consider an observable of the first kind as a set of particles, each value of the observable being one particle. The function p then gives a probability distribution of the particles in the phase-space E_{n-1}. The definition [7.5] gives a property of the set of particles. It is possible to allot this property to the individual particles on a quasi-additive basis, in the following manner. Consider a variation of the probability distribution which does not change p or its derivatives on the boundary of the region of integration. We then have

$$\delta F = \int \frac{\delta f}{\delta p}\, \delta p \, d\tau$$

by the usual calculus-of-variations treatment, where $\delta f/\delta p$ is the Hamiltonian derivative. Thus a small increase δp in p produces an increase $(\delta f/\delta p)\,\delta p$ in F per unit volume. It is therefore natural to define $\delta f/\delta p$ as the 'F per particle per unit volume'. Hence

[8.1] DEFINITION. *The total particle-F is* $F_1 = \int p \frac{\delta f}{\delta p}\, d\tau$.

The fact that $F \neq F_1$ can then be attributed to further F residing in the field, and this forms a definition of the concept of field corresponding to the concept of particle which we have used. Thus

[8.2] DEFINITION. *The total field-F is* $F_2 = F - F_1$.

If, however, F is a Hamiltonian property and is scale-free and reducible, F_1 is easily found in terms of F. We must now consider a

change in scale. This changes the volume metric of the space, as well as the coordinates, so that

$$\delta F = \int \left(\frac{\partial f}{\partial p}\,\delta p + \frac{\partial f}{\partial p_\mu}\,\delta p_\mu + \ldots\right) d\tau + \int f\,\delta(d\tau).$$

Since Ψ is scale-free, we can consider the scale-change

$$\sigma' = \sigma/(1+\epsilon).$$

Then

$$d\tau' = (1+\epsilon)^k\, d\tau$$

if the particles are k-dimensional. Hence, by the usual treatment and substitution,

$$\delta F = \int \frac{\delta f}{\delta p}\,\delta p\, d\tau + k\epsilon F,$$

for ϵ small. This last step is justified provided that this variation does not change p and its derivatives on the boundary. Now Ψ is reducible; hence, by [6.8],

$$\delta p = -k\epsilon p,$$

and so, by [8.1],

$$\delta F = k\epsilon(F - F_1).$$

On the other hand, F is a Hamiltonian property so that

$$\delta f = l\epsilon f \quad \text{(for some } l\text{)},$$

and

$$\delta F = \delta \int f\, d\tau = (l+k)\epsilon F.$$

Comparing these results gives

[8.3] THEOREM. $$F = -\frac{k}{l} F_1.$$

Hence also

[8.4] THEOREM. $$F_2 = -\frac{l+k}{l} F_1.$$

These are the results given by Eddington [(**3**) (5.51)]. The restrictions under which they have been proved are that Ψ should be scale-free and reducible, and F should be a Hamiltonian property. Only the scale-free condition was stated by Eddington.

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A NOTE ON ENTIRE FUNCTIONS (DEFINED BY DIRICHLET'S SERIES) OF PERFECTLY REGULAR GROWTH

By Q. I. RAHMAN (*Aligarh*)

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1. Consider the Dirichlet's series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n},$$

where $\lambda_{n+1} > \lambda_n, \quad \lambda_1 \geqslant 0, \quad \lim_{n\to\infty} \lambda_n = \infty, \quad s = \sigma+it,$

and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \tag{1}$$

It defines in its half-plane of convergence a holomorphic function. Let σ_c and σ_a be respectively the abscissa of convergence and the abscissa of absolute convergence of $f(s)$.

Let $\mu(\sigma)$ be the maximum of $|a_n|e^{\sigma\lambda_n}$ $(n = 1, 2,...)$, and $M(\sigma)$ the l.u.b. of $|f(\sigma+it)|$ $(-\infty < t < \infty)$, where σ is a constant smaller than σ_a. If $\sigma_c = \infty$, $\sigma_a = \infty$, $f(s)$ defines an *entire function*. Let $\lambda_{\nu(\sigma)}$ be the λ_n corresponding to the maximum term of the series for $\operatorname{re} s = \sigma$. Then evidently $\lambda_{\nu(\sigma)}$ is a non-decreasing function of σ.

Let $L(x)$ be a 'slowly changing' function: that is, $L(x) > 0$ and is continuous for $x > e^{\sigma_0}$ and $L(cx) \sim L(x)$ as $x \to \infty$, for every constant $c > 0$. I compare $\log M(\sigma)$ with the function $e^{\rho\sigma}L(e^{\sigma})$ (ρ being the Ritt order of the entire function $f(s)$) and establish the following result.

2. When $0 < \rho < \infty$, let

$$\lim_{\sigma\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{\log M(\sigma)}{e^{\rho\sigma}L(e^{\sigma})} = \begin{Bmatrix}T,\\ \tau;\end{Bmatrix} \qquad \lim_{\sigma\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{\lambda_{\nu(\sigma)}}{e^{\rho\sigma}L(e^{\sigma})} = \begin{Bmatrix}\gamma,\\ \delta.\end{Bmatrix}$$

Theorem. (i) *If* $0 < \tau \leqslant T < \infty$, *then* $0 < \delta \leqslant \gamma < \infty$ *and conversely.*

(ii) *If* (i) *holds, then*

$$\frac{1}{\rho e^{\kappa}} < \lim_{\sigma\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{\log M(\sigma)}{\lambda_{\nu(\sigma)}} < \frac{e^{\kappa}}{\rho}, \tag{2}$$

where $x = \kappa$ *is that root of the equation* $eTx = e^{x}\tau - eT$ *which lies in the interval* $(1, \infty)$.

(iii) *If* $\log M(\sigma) \sim Te^{\rho\sigma}L(e^{\sigma})$ $(0 < T < \infty)$, *then*

$$\lambda_{\nu(\sigma)} \sim \rho T e^{\rho\sigma} L(e^{\sigma}),$$

and conversely.

Quart. J. Math. Oxford (2), 6 (1955), 173-5.

Proof (i) If $a \geqslant 0$, $\gamma < \infty$, then, if

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0,$$

we have [(**2**) 73]

$$\log M\left(\sigma+\frac{a}{\rho}\right) \sim \log\mu\left(\sigma+\frac{a}{\rho}\right)$$

$$= O(1)+\left[\int\limits_{\sigma_0}^{\sigma} + \int\limits_{\sigma}^{\sigma+a/\rho}\right]\lambda_{\nu(t)}\,dt, \quad \text{by } (\mathbf{2})\ 67,$$

$$< O(1)+(\gamma+\epsilon)\int\limits_{\sigma_0}^{\sigma} e^{\rho\sigma}L(e^{\sigma})\,d\sigma+\frac{a\lambda_{\nu(\sigma+a/\rho)}}{\rho}$$

$$= O(1)+(\gamma+\epsilon)\int\limits_{e^{\sigma_0}}^{e^{\sigma}} x^{\rho-1}L(x)\,dx+\frac{a\lambda_{\nu(\sigma+a/\rho)}}{\rho}$$

$$\sim (\gamma+\epsilon)\frac{e^{\rho\sigma}}{\rho}L(e^{\sigma})+\frac{a\lambda_{\nu(\sigma+a/\rho)}}{\rho}, \quad \text{by } (\mathbf{1}) \text{ Lemma 5}.$$

Hence $$\rho e^{a}T \leqslant \gamma+\gamma a e^{a}, \qquad \rho e^{a}\tau \leqslant \gamma+\delta a e^{a}, \tag{3}$$

which holds also when $\gamma = \infty$. Similarly we get

$$\rho e^{a}T \geqslant \delta+\gamma a, \qquad \rho e^{a}\tau \geqslant \delta(1+a). \tag{4}$$

Suppose now that $0 < \tau \leqslant T < \infty$. From (4) we get $\gamma < \infty$. Further, $\delta > 0$. For, if $\delta = 0$, we get from (3) $\tau \leqslant \gamma/\rho e^{a}$, and, since a is arbitrary, it follows that $\tau = 0$. Hence we have a contradiction, and so $\delta > 0$.

If $0 < \delta \leqslant \gamma < \infty$, then from (3) we have $T < \infty$ and, from (4), $\tau > 0$.

(ii) Take $$a = \frac{\gamma-\delta}{\gamma}, \qquad \gamma \leqslant \rho T\exp\left(1-\frac{\delta}{\gamma}\right) < \rho Te,$$

and hence, from (3), $$\rho e^{a}\tau < \rho Te+\delta e^{a}a.$$

Consider now the equation

$$eTx = e^{x}\tau - eT.$$

It has one and only one root in the interval $(1,\infty)$. Let it be κ. Then taking $a = \kappa$ we get

$$\rho(e^{\kappa}\tau - Te) < \delta e^{\kappa}\kappa,$$

i.e. $$\rho Te\kappa < \delta e^{\kappa}\kappa, \qquad \delta > \frac{\rho e T}{e^{\kappa}}.$$

Hence $$\frac{1}{\rho e^{\kappa}} < \frac{\delta}{\gamma\rho} \leqslant \lim_{\sigma\to\infty}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log M(\sigma)}{\lambda_{\nu(\sigma)}} \leqslant \frac{\gamma}{\delta\rho} < \frac{e^{\kappa}}{\rho}. \tag{5}$$

(iii) From (3) and (4) we get

$$\delta \leqslant \rho T \leqslant \gamma, \qquad \delta \leqslant \rho\tau \leqslant \gamma,$$

and hence, if $\delta = \gamma$, $T = \tau = \gamma/\rho$. Suppose now that $T = \tau$, which we may take to be unity without loss of generality. I shall show that $\gamma = \delta$. If $0 < \eta < 1$, then, if

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0,$$

we have
$$\log M(\sigma) \sim I(\sigma) = \int_{\sigma_0}^{\sigma} \lambda_{\nu(t)}\,dt,$$

$$\begin{aligned}
\lambda_{\nu(\sigma)}\eta < \int_{\sigma}^{\sigma+\eta} \lambda_{\nu(t)}\,dt &= I(\sigma+\eta)-I(\sigma) \\
&= e^{\rho(\sigma+\eta)}L(e^{\sigma+\eta})-e^{\rho\sigma}L(e^{\sigma})+o\{e^{\rho\sigma}L(e^{\sigma})\} \\
&= e^{\rho\sigma}\{1+\rho\eta+O(\eta^2)\}\{1+o(1)\}L(e^{\sigma})-e^{\rho\sigma}L(e^{\sigma})+ \\
&\qquad\qquad +o\{e^{\rho\sigma}L(e^{\sigma})\} \\
&= e^{\rho\sigma}L(e^{\sigma})\{\rho\eta+H\eta^2+o(1)\}.
\end{aligned}$$

Hence
$$\limsup_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{e^{\rho\sigma}L(e^{\sigma})} < \rho+H\eta,$$

where H is a constant. Since η is arbitrary, we get

$$\limsup_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{e^{\rho\sigma}L(e^{\sigma})} \leqslant \rho.$$

By considering the integral $I(\sigma)-I(\sigma-\eta)$, we get

$$\liminf_{\sigma\to\infty} \frac{\lambda_{\nu(\sigma)}}{e^{\rho\sigma}L(e^{\sigma})} \geqslant \rho,$$

and hence
$$\lambda_{\nu(\sigma)} \sim \rho e^{\rho\sigma}L(e^{\sigma}).$$

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THE MAXIMUM MODULUS OF AN INTEGRAL FUNCTION OF AN INTEGRAL FUNCTION

By J. CLUNIE (*Keele*)

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1. WHEN considering the problem of what conditions two integral functions $g(z)$ and $h(z)$ must satisfy in order that $g\{h(z)\}$ be of finite order, Pólya (**2**) used the following result.

Suppose $f(z)$, $g(z)$, $h(z)$ to be three integral functions connected by the relation

$$f(z) = g\{h(z)\}.$$

Suppose further that $$h(0) = 0.$$

Let $F(r)$, $G(r)$, $H(r)$ denote the maximum moduli of $f(z)$, $g(z)$, $h(z)$ respectively in the circle $|z| \leqslant r$. Then there is a finite number c, greater than zero and less than 1, *and such that*

$$F(r) \geqslant G\{cH(\tfrac{1}{2}r)\}.$$

His method of proof was based on a theorem of Bohr which left the constant c undetermined.

In this paper I shall prove the theorem:

THEOREM. *Outside an exceptional set of intervals of r, within which the variation of* $\log r$ *is finite, we have, for sufficiently large r,*

$$F\{r\exp(k_1 N^{-\frac{1}{2}} \log^{\frac{1}{2}} N)\} \geqslant G\{H(r)\},$$

where $N = N(r)$ is the central index of $h(z)$ for $|z| = r$ and k_1 is a constant depending on $h(z)$.

If r lies in an exceptional interval and is sufficiently large, then

$$F[r\exp\{k_1 N_1^{-\frac{1}{2}} \log^{\frac{1}{2}} N_1 - \mu(r_1)/\log N_1\}] \geqslant G[H\{r\exp(-\mu(r_1)/\log N_1)\}],$$

where k_1 is the same constant as before, $\mu(x)$ is a function such that, as $x \to \infty$,

$$\limsup \mu(x) \leqslant 1,$$

and $N_1 = N(r_1)$ is the central index of $h(z)$ for $|z| = r_1$, with

$$r = r_1 \exp\{\mu(r_1)/\log N_1\}.$$

2. The proof of the theorem requires the following lemma [(1) 38].

Lemma. *Any integral function $h(z)$, outside a set of intervals within which the variation of* $\log|z|$ *is finite, satisfies*

$$h(ze^{\tau}) = e^{N\tau}h(z)\{1+\omega(\tau)\}, \qquad N = N(r),$$

at points where $|h(z)| = H(|z|)$ *with* $|\omega(\tau)| < k_2 N^{-\frac{1}{2}}\log^{\frac{1}{2}} N$ *provided that* $|\tau| < 2/N$; k_2 *is a constant depending on* $h(z)$.

Values of $|z|$ for which the above lemma is true are called *ordinary* and other values *exceptional*.

I shall first of all deal with the case when $|z|$ is an ordinary value. It follows that for such a value, with $|z| = r$ and $N(r) > 1$,

$$h(ze^{\tau}) = e^{N\tau+i\phi}H(r)\{1+\omega(\tau)\},$$

where τ and $\omega(\tau)$ satisfy the conditions of the lemma and $\phi = \arg h(z)$, $|h(z)| = H(r)$. Let $H(r)e^{i\psi}$ be a point such that

$$|g\{H(r)e^{i\psi}\}| = G\{H(r)\}.$$

For $\tau = i(\psi-\phi)/N = \tau_0$, say, we have

$$e^{N\tau+i\phi}H(r) = H(r)e^{i\psi}.$$

Consider now

$$h(ze^{\tau})-e^{N\tau_0+i\phi}H(r) = e^{N\tau+i\phi}H(r)\{1-e^{N(\tau_0-\tau)}\}+e^{N\tau+i\phi}H(r)\omega(\tau).$$

When τ describes a small circle about τ_0 so that

$$\tau-\tau_0 = k_1 N^{-\frac{3}{2}}\log^{\frac{1}{2}} N e^{i\theta} \quad (0 \leqslant \theta \leqslant 2\pi),$$

then
$$|e^{N\tau+i\phi}H(r)\{1-e^{N(\tau_0-\tau)}\}| \sim |e^{N\tau}|H(r)k_1 N^{-\frac{1}{2}}\log^{\frac{1}{2}} N.$$

On this circle $|\tau| = N\{1+o(1)\}$, which means that, if r is large enough, then

$$|\omega(\tau)| < k_2 N^{-\frac{1}{2}}\log^{\frac{1}{2}} N.$$

Hence we can choose $k_1 > k_2$ and then it follows that on the circle, provided that r is large enough,

$$|e^{N\tau+i\phi}H(r)\{1-e^{N(\tau_0-\tau)}\}| > |e^{N\tau+i\phi}H(r)\omega(\tau)|,$$

and so, by Rouché's theorem, $h(ze^{\tau})$ takes the value $H(r)e^{i\psi}$ somewhere in the circle. Hence

$$F\{r\exp(k_1 N^{-\frac{3}{2}}\log^{\frac{1}{2}} N)\} \geqslant G\{H(r)\}.$$

We now consider the case when $|z| = r$ is an exceptional value. Take r so large that the previous result holds for r_1, the largest ordinary value not exceeding r. The variation of $\log r$ in the exceptional intervals for $r \geqslant R$ does not exceed $\lambda(R)/\log N(R)$, where $\lambda(R) \to 1$ as $R \to \infty$ [(2) 37]. Hence

$$r = r_1\exp\{\mu(r_1)/\log N(r_1)\},$$

where, as $x \to \infty$, $\limsup \mu(x) \leqslant 1$, and so

$$F[r\exp\{k_1 N_1^{-\frac{3}{4}} \log^{\frac{3}{4}} N_1 - \mu(r_1)/\log N_1\}] \geqslant G[H\{r\exp(-\mu(r_1)/\log N_1)\}].$$

3. It is of interest to see how the above theorem compares with the similar one that can be deduced from Theorem 29 of Valiron's *Lectures*. By means of the same reasoning as used above we can show that, provided that r is large enough, for ordinary values

$$F\{r\exp(k_3 N^{-17/16})\} \geqslant G\{H(r)\}$$

and for exceptional values

$$F\{r\exp(k_3 N_1^{-17/16} - \nu(r_1)N_1^{-1/12})\} \geqslant G[H\{r\exp(-\nu(r_1)N_1^{-1/12})\}],$$

where k_3 is a constant depending on $h(z)$ and $\nu(x)$ is a function such that, as $x \to \infty$, $\limsup \nu(x) \leqslant \frac{11}{12}$, with $r = r_1 \exp\{\nu(r_1)N_1^{-1/12}\}$.

The theorem, and the result obtained above depending on Valiron's Theorem 29, can both be expressed in the simpler, but weaker form

$$F[r\{1+o(1)\}] \geqslant G\{H(r)\}.$$

4. When $g(z)$ and $h(z)$ are complete integral functions, i.e. not polynomials, then, as pointed out by Pólya, $g(z)$ is necessarily of zero order if $f(z)$ is of finite order. In addition the following results can be deduced from the theorem and the obvious inequality $F(r) \leqslant G\{H(r)\}$ if $f(z)$ is of finite order:

(i) when $h(z)$ is of lower non-zero order λ and upper order μ, then, for some finite β, $\log G(r)/(\log r)^{\beta} \to 0$; if B is the lower bound of such numbers β, then $1 \leqslant B < \infty$, and $f(z)$ is of lower order not less than λB and upper order not greater than μB;

(ii) when $f(z)$ and $h(z)$ are of the same order, then $f(z)$ is of infinite type if $h(z)$ is of finite non-zero type. ('Type' is assumed to have its usual meaning.)

These two conclusions also follow from Pólya's result.

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INDEFINITELY ISOMETRIC LINEAR OPERATORS IN A REFLEXIVE BANACH SPACE

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1. LET V be a reflexive Banach space over the real field, ϕ a non-zero continuous linear functional on V, and e a point of V such that $\phi(e) = 1$. The linear operator P defined by

$$Px = \phi(x)e \quad (x \in V)$$

is a projection, i.e. $P^2 = P$. Let

$$Q = I - P,$$

where I is the identity operator. Plainly, Q is a projection orthogonal to P, i.e. $Q^2 = Q$ and $PQ = QP = 0$. Let ν be a positive real number. I introduce into V the 'indefinite norm' q, given by

$$q(x) = \|Qx\|^\nu - |\phi(x)|^\nu, \tag{1}$$

and consider linear operators T in V that are 'isometric' with respect to q, i.e. those that satisfy

$$q(Tx) = q(x) \quad (x \in V).$$

More generally, I consider linear operators T that are 'bounded' with respect to q, in the sense that there exists a positive constant M for which

$$q(Tx) \leqslant Mq(x) \quad (x \in V).$$

Except for a certain homogeneity $[q(\lambda x) = |\lambda|^\nu q(x)]$, q does not in general possess norm-like properties. Moreover, q is, on the face of it, a rather artificial and unnatural object. I therefore feel obliged to give some justification for considering the present problem and for using the term 'indefinite norm' for q.

As a special case, we may consider the space $(l^{(p)})$, with $1 < p < \infty$, of all real sequences

$$x = \{\xi_k\}_{k=1}^{\infty} \text{ with } \sum_{k=1}^{\infty} |\xi_k|^p < \infty,$$

and take

$$q_p(x) = -|\xi_1|^p + \sum_{k=2}^{\infty} |\xi_k|^p.$$

This is of the form (1) if we take

$$e = \{\delta_{1k}\} \quad (\delta_{11} = 1,\ \delta_{1k} = 0\ (k \neq 1)), \quad \phi(x) = \xi_1, \quad \nu = p.$$

The use of the term 'indefinite norm' for the functional q_p seems not unnatural. My only justification for making public a study of linear operators bounded or isometric with respect to an indefinite norm is that I am able to establish some rather precise, and I think interesting, properties of their spectra.

The main idea of the present article is derived from the work of M. G. Krein and M. A. Rutman on Lorentz transformations of a sequential Hilbert space (**4**). Krein and Rutman consider the indefinite bilinear form in $(l^{(2)})$ defined by

$$[x,y] = \xi_1 \eta_1 - \sum_{k=2}^{\infty} \xi_k \eta_k,$$

where $x = \{\xi_k\}$, $y = \{\eta_k\}$. They study the spectra of (1–1) mappings T of $(l^{(2)})$ on to itself that satisfy $[Tx, Ty] = [x,y]$ for all x, $y \in (l^{(2)})$, by applying their theory of linear operators that map the interior of a cone into itself. The results of Krein and Rutman in this field are related to those of L. Pontrjagin (**5**) on operators in complex $(l^{(2)})$ that are Hermitian with respect to a similar (though more general) bilinear form: see also the paper (**3**) by I. S. Iohvidov.

After some preliminaries, I consider a linear operator T that is bounded with respect to an indefinite norm. I show that T is bounded with respect to the usual norm and that, if ρ is its spectral radius, then either ρ or $-\rho$ is a characteristic number. Since no condition of complete continuity has been imposed on T, this does not seem to be a trivial result. In § 4 I slightly restrict the generality of V and q and suppose that T is isometric with respect to q. For such isometric T, much more precise properties of the spectrum are established. These properties are set out in Theorem 2.

The proofs depend on a theorem concerning endomorphisms of a partially ordered vector space that is proved in (**1**). An account of the terminology and elementary properties of partially ordered vector spaces is to be found in (**2**).

2. Preliminary lemmas

We note first that each $x \in V$ may be written in the form

$$x = \phi(x)e + Qx, \tag{2}$$

and that $Qe = 0$, $Q^*\phi = 0$. We denote by V^+ the set of all $x \in V$ that satisfy

$$\phi(x) \geqslant \|Qx\|.$$

LEMMA 1. *V^+ is a closed cone of which e is an interior point. More generally, x is an interior point of V^+ if and only if $\phi(x) > \|Qx\|$.*

Proof. If $x_i \in V^+$ $(i = 1, 2)$, then

$$\phi(x_1+x_2) = \phi(x_1)+\phi(x_2) \geqslant \|Qx_1\|+\|Qx_2\| \geqslant \|Q(x_1+x_2)\|,$$

and therefore $x_1+x_2 \in V^+$. Obviously, if $x \in V^+$ and $\alpha \geqslant 0$, then $\alpha x \in V^+$. Using (2), it is equally easy to prove that, if x, $-x \in V^+$, then $x = 0$. That V is closed is an obvious consequence of the continuity of ϕ and Q. For the same reason, if $\phi(x) > \|Qx\|$, then x is an interior point of V^+. In particular, e is an interior point of V^+.

On the other hand, if $\phi(x) = \|Qx\|$, then, for every $\epsilon > 0$,

$$\phi(x-\epsilon e) < \|Qx\| = \|Q(x-\epsilon e)\|,$$

and therefore x is not an interior point of V^+. This completes the proof of the lemma.

With V^+ as the set of non-negative elements, V is a partially ordered vector space with an order unit. I write, as usual, $x \geqslant y$ to denote that $(x-y) \in V^+$. The set of linear functionals non-negative on V^+ is denoted by V^{*+}. Since V^+ has interior points, each $\psi \in V^{*+}$ is continuous; also V^{*+} is a closed cone in V^*. Let K denote the set of all continuous linear functionals $\psi \in V^*$ such that $\psi(e) \geqslant \|Q^*\psi\|$, Q^* being the adjoint of Q [i.e. $(Q^*\psi)(x) = \psi(Qx)$, $x \in V$].

LEMMA 2. (i) $K \subset V^{*+}$.

(ii) *Each $\psi \in V^*$ with $\psi(e) > \|Q^*\psi\|$ is an interior point of K and therefore of V^{*+}. In particular, ϕ is an interior point of V^{*+}.*

(iii) *If $\|Q\| = 1$, then $K = V^{*+}$.*

Proof. (i) Let $\psi \in K$ and $x \in V^+$. Then, using (2), we have

$$\begin{aligned}\psi(x) &= \phi(x)\psi(e)+\psi(Qx)\\ &= \phi(x)\psi(e)+\psi(Q^2x)\\ &= \phi(x)\psi(e)+(Q^*\psi)(Qx)\\ &\geqslant \phi(x)\psi(e)-\|Q^*\psi\|.\|Qx\|\\ &\geqslant 0.\end{aligned}$$

(ii) Since each $\psi \in V^*$ may be written in the form $\psi = \psi(e)\phi+Q^*\psi$, and since $Q^*\phi = 0$, it follows just as in the proof of Lemma 1 that K is a cone in V^* having as interior points all those $\psi \in V^*$ with $\psi(e) > \|Q^*\psi\|$, including, in particular, ϕ.

(iii) Suppose now that $\|Q\| = 1$, and that $\psi \in V^{*+}$. Since V is reflexive, we may choose $y \in V$ with $\|y\| = 1$ and $(Q^*\psi)(y) = \|Q^*\psi\|$. Let

$$z = \|Qy\|e-Qy.$$

We have $Qz = -Qy$, $\phi(z) = \|Qy\| = \|Qz\|$. This shows that $z \geqslant 0$ and therefore $\psi(z) \geqslant 0$. Now

$$0 \leqslant \psi(z) = \|Qy\|\psi(e) - \psi(Qy) = \|Qy\|\psi(e) - \|Q^*\psi\|.$$

Since $\|Q\| = 1$, we have $\|Qy\| \leqslant 1$ and therefore $\psi(e) \geqslant \|Q^*\psi\|$; i.e. $\psi \in K$.

LEMMA 3. *The given norm in V is equivalent to the norm $|\,.\,|$ defined by*

$$|x| = \inf[\xi\colon -\xi e \leqslant x \leqslant \xi e].$$

Proof. Since V^+ has interior points, there is a constant η such that $|x| \leqslant \eta\|x\|$, $x \in V$. An inequality in the opposite direction is a simple consequence of the fact that V^{*+} has interior points.

In the present case a direct proof is available, which the author owes to the referee. It is easily verified that $|x| = |\phi(x)| + \|Qx\|$. Thus $|x| \leqslant (\|\phi\| + \|Q\|)\|x\|$. On the other hand, since $Px = \phi(x)e$ and

$$Px + Qx = x,$$

we have

$$\begin{aligned} |x| &= \|e\|^{-1}.\|Px\| + \|Qx\| \\ &\geqslant \min(1, \|e\|^{-1})(\|Px\| + \|x - Px\|) \\ &\geqslant \min(1, \|e\|^{-1})\|x\|. \end{aligned}$$

We say that the closed unit sphere in V is *strictly convex* if the equations $\|x_i\| = 1$, $i = 1, 2$, imply that $\|\tfrac{1}{2}(x_1 + x_2)\| < 1$ unless $x_1 = x_2$.

LEMMA 4. *Let $\psi \in V^*$ be non-zero and satisfy $\psi(e) = \|Q^*\psi\|$. If the closed unit sphere in V is strictly convex and $\psi(u_i) = 0$, $\phi(u_i) = \|Qu_i\|$ $(i = 1, 2)$, then u_1, u_2 are linearly dependent.*

Proof. If either of u_1, u_2 is zero, there is nothing to prove. Suppose then that $u_i \neq 0$ $(i = 1, 2)$. This implies that also $Qu_i \neq 0$; for, if $Qu_i = 0$, then $u_i = \phi(u_i)e$. But, since ψ is a non-zero element of V^{*+}, we have $\psi(e) \neq 0$, and we therefore obtain $\phi(u_i) = 0$, $u_i = 0$, contrary to hypothesis. We have

$$\begin{aligned} 0 = \psi(u_i) &= \phi(u_i)\psi(e) + \psi(Qu_i) \\ &= \phi(u_i)\psi(e) + \psi(Q^2u_i) \\ &= \phi(u_i)\psi(e) + (Q^*\psi)(Qu_i). \end{aligned}$$

It follows that

$$(Q^*\psi)(-Qu_i) = \phi(u_i)\psi(e) = \|Q^*\psi\|.\|Qu_i\|,$$

i.e. the functional $Q^*\psi$ attains its bound at $-(Qu_i)/\|Qu_i\|$. But, owing to the strict convexity of the closed unit sphere, a linear functional attains its bound at exactly one point of the closed unit sphere. Consequently,

$$(Qu_1)/\|Qu_1\| = (Qu_2)/\|Qu_2\|.$$

Since $\phi(u_i) = \|Qu_i\|$, it follows from the equations $u_i = \phi(u_i)e + Qu_i$ that $u_1/\|Qu_1\| = u_2/\|Qu_2\|$.

3. Indefinitely bounded linear operators

Definition. An additive and real-homogeneous mapping T of V into itself is called an *indefinitely bounded linear operator* if there exists a constant $M > 0$, called an *indefinite bound*, such that

$$q(Tx) \leqslant Mq(x) \quad (x \in V).$$

LEMMA 5. *If T is an indefinitely bounded linear operator, then T is bounded and either T or $-T$ is an endomorphism of the partially ordered vector space V.*

Proof. If $x \geqslant 0$, then $\phi(x) \geqslant \|Qx\|$, $q(x) \leqslant 0$, and therefore $q(Tx) \leqslant 0$. This implies that $|\phi(Tx)| \geqslant \|QTx\|$; hence $Tx \geqslant 0$ or $Tx \leqslant 0$ according as $\phi(Tx) \geqslant 0$ or $\phi(Tx) \leqslant 0$. In particular, $\phi(Tx) = 0$ implies $Tx = 0$.

Suppose that u, v are order units of V and that $Tu > 0$ while $Tv < 0$. Then $\phi(Tu) > 0$ and $\phi(Tv) < 0$; therefore, we may choose positive numbers λ, μ such that $\phi(T(\lambda u + \mu v)) = 0$. As we have seen, this implies that $T(\lambda u + \mu v) = 0$. But $\lambda u + \mu v$ is an order unit of V and therefore

$$q(T(\lambda u + \mu v)) \leqslant Mq(\lambda u + \mu v) < 0.$$

We have now proved that either T or $-T$ maps the order units of V into V^+. Since $x + \epsilon e$ is an order unit whenever $x \in V^+$ and $\epsilon > 0$, it follows that either T or $-T$ is an endomorphism of the partially ordered vector space V.

The boundedness of T with respect to the usual norm in V is now an immediate consequence of Lemma 3.

Definition. An indefinitely bounded linear operator T is said to be of the *first* or *second kind* according as T or $-T$ is an endomorphism of V.

We recall that the spectral radius of a bounded linear operator T is the maximum modulus of the complex numbers in the spectrum of T, and is given by the formula

$$\rho = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

The following lemma, which strengthens a result given by the author, is due to the referee.

LEMMA 6. *If T is an indefinitely bounded linear operator, then*

$$\rho \geqslant M^{1/\nu},$$

where ρ is the spectral radius of T and M is an indefinite bound.

Proof. Since M (>0) is an indefinite bound and $q(e) = -1$, we have $q(T^n e) \leqslant -M^n$; i.e.

$$\|QT^n e\|^\nu - |\phi(T^n e)|^\nu \leqslant -M^n.$$

Thus $|\phi(T^n e)| \geqslant M^{n/\nu}$, $\|\phi\|\,\|T^n\|\,\|e\| \geqslant M^{n/\nu}$, $\|T^n\|^{1/n} \geqslant M^{1/\nu}\|\phi\|^{-1/n}\|e\|^{-1/n}$. Letting n tend to infinity, we obtain $\rho \geqslant M^{1/\nu}$.

THEOREM 1. *Let T be an indefinitely bounded linear operator of the first kind. Then T is a bounded linear operator with a non-zero spectral radius ρ, which is a characteristic number of T and of T^*; in fact, there exist non-zero elements $u \in V^+$, $\psi \in V^{*+}$ such that*

$$Tu = \rho u, \qquad T^*\psi = \rho\psi.$$

Proof. We have proved in the last lemma that $\rho \neq 0$. Since T is an indefinitely bounded linear operator of the first kind, it is an endomorphism of the partially ordered vector space V. V being a reflexive space such that V^+ and V^{*+} are closed cones with interior points, the theorem follows at once from Theorem 3 in (**1**).

4. Indefinitely isometric linear operators

Definition. A linear operator T mapping V (1–1) on to itself is called an *indefinitely isometric* linear operator if

$$q(Tx) = q(x) \quad (x \in V).$$

An indefinitely isometric linear operator is, of course, indefinitely bounded, as is also its inverse T^{-1}; thus Theorem 1 is applicable to T and to T^{-1}. Moreover it is plain that, if T is of the first kind, then so is T^{-1}.

ASSUMPTION. *Throughout the remainder of this article, I shall suppose that the closed unit sphere in V is strictly convex and that $\|Q\| = 1$.*

I now state the main theorem.

THEOREM 2. *Let T be an indefinitely isometric linear operator of the first kind in V. Then T, T^{-1} are bounded linear operators with spectral radii ρ, σ satisfying $\sigma^{-1} \leqslant 1 \leqslant \rho$, and there are two possible cases.*

Case 1: $\sigma^{-1} = \rho$. *The spectrum of T lies on the circumference of the unit circle and 1 is a characteristic number of T and of T^*.*

Case 2: $\sigma^{-1} < \rho$. *The numbers ρ, σ^{-1} are characteristic numbers of T and of T^*, and the rest of the spectrum lies on the circumference of the unit circle. In more detail, there exist non-zero elements $u_1, u_2 \in V^+$, $\psi_1, \psi_2 \in V^{*+}$ such that*

$$Tu_1 = \rho u_1, \qquad Tu_2 = \sigma^{-1}u_2, \qquad T^*\psi_1 = \rho\psi_1, \qquad T^*\psi_2 = \sigma^{-1}\psi_2,$$

$$\psi_i(u_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

If W denotes the intersection of the null manifolds of ψ_1 and ψ_2, then each $x \in V$ is expressible in the form

$$x = \psi_1(x)u_1+\psi_2(x)u_2+w \quad (w \in W);$$

T maps W into itself and, when regarded as an operator in W, its spectrum lies on the circumference of the unit circle.

If either ρ or σ^{-1} is different from 1, *then it is a simple characteristic number.*

Proof. Applying Lemma 6 with $M = 1$ to both T and T^{-1}, we have $\rho \geqslant 1$ and $\sigma \geqslant 1$. Case 1 is an immediate consequence of Theorem 1. Suppose now that $\sigma^{-1} < \rho$. Applying Theorem 1, we know that there exist non-zero elements u_1, $u_2 \in V^+$, ψ_1, $\psi_2 \in V^{*+}$ such that

$$Tu_1 = \rho u_1, \qquad T^*\psi_1 = \rho\psi_1, \qquad T^{-1}u_2 = \sigma u_2, \qquad T^{-1*}\psi_2 = \sigma\psi_2.$$

The last two equations may be rewritten in the form

$$Tu_2 = \sigma^{-1}u_2, \qquad T^*\psi_2 = \sigma^{-1}\psi_2.$$

We have $$\rho\psi_1(u_2) = T^*\psi_1(u_2) = \psi_1(Tu_2) = \sigma^{-1}\psi_1(u_2),$$

and, since $\rho \neq \sigma^{-1}$, it follows that $\psi_1(u_2) = 0$. Similarly, $\psi_2(u_1) = 0$. We note also that u_1, u_2 are linearly independent.

Let $\mathfrak{M}_i$ denote the null manifold of ψ_i $(i = 1, 2)$. Since $\mathfrak{M}_i$ is a proper ideal of the partially ordered vector space V and a proper ideal does not contain any order units, we see that u_1, u_2 are frontier points of V^+. Similarly ψ_1, ψ_2 are frontier points of V^{*+}. This proves that

$$\phi(u_i) = \|Qu_i\|, \qquad \psi_i(e) = \|Q^*\psi_i\| \quad (i = 1, 2).$$

By Lemma 4, we see that, if $x \in \mathfrak{M}_1$ and $\phi(x) = \|Qx\|$, then x is linearly dependent on u_2. In particular it follows that $u_1 \notin \mathfrak{M}_1$, and similarly $u_2 \notin \mathfrak{M}_2$. By a normalization, we may therefore arrange that

$$\psi_1(u_1) = \psi_2(u_2) = 1.$$

Let $W = \mathfrak{M}_1 \cap \mathfrak{M}_2$. Each $x \in V$ is evidently expressible in the form

$$x = \psi_1(x)u_1+\psi_2(x)u_2+w \quad (w \in W).$$

Moreover, since ψ_i is a characteristic vector of T^*, we have $T\mathfrak{M}_i \subset \mathfrak{M}_i$ $(i = 1, 2)$, and therefore $TW \subset W$. Similarly $T^{-1}W \subset W$.

We prove next that $q(x) \geqslant 0$ for all $x \in W$, with equality if and only if $x = 0$. In fact, if $q(x) < 0$, then either x or $-x$ is an order unit; but this is impossible for an element of W. If $q(x) = 0$, then, by Lemma 4, x is linearly dependent on u_1 and on u_2. This is impossible unless $x = 0$.

We prove that there exists a positive number γ such that

$$\{q(x)\}^{1/\nu} \geqslant \gamma\|x\| \quad (x \in W). \tag{3}$$

Since $\{q(x)\}^{1/\nu}$ is positive homogeneous, it is enough to prove that

$$\inf_{x\in W, \|x\|=1} \{q(x)\}^{1/\nu} = \gamma > 0. \tag{4}$$

Suppose, on the contrary, that

$$\inf_{x\in W, \|x\|=1} q(x) = 0.$$

Then there exists a sequence $\{x_n\}$ ($x_n \in W$) such that $\|x_n\| = 1$, while $\lim_{n\to\infty} q(x_n) = 0$. The closed unit sphere in a reflexive space being sequentially compact in the weak topology, there exists $x \in V$ with $\|x\| \leqslant 1$ and a subsequence $\{x_{n_k}\}$ converging weakly to x. In particular,

$$\lim_{k\to\infty} \phi(x_{n_k}) = \phi(x). \tag{5}$$

Since $\lim_{k\to\infty} q(x_{n_k}) = 0$, we have

$$\lim_{k\to\infty} \|Qx_{n_k}\| = |\phi(x)|. \tag{6}$$

Now closed spheres are weakly closed and $\{Qx_{n_k}\}$ converges weakly to Qx. It follows that

$$\|Qx\| \leqslant \lim_{k\to\infty} \|Qx_{n_k}\| = |\phi(x)|,$$

and therefore $q(x) \leqslant 0$. But W, being a closed subspace, is weakly closed and therefore $x \in W$. Since $q(x) \leqslant 0$, this implies that $x = 0$. It now follows from (5) and (6) that $\lim_{k\to\infty} \phi(x_{n_k}) = 0$ and $\lim_{k\to\infty} \|Qx_{n_k}\| = 0$. Since

$$x_{n_k} = \phi(x_{n_k})e + Qx_{n_k},$$

this implies that $\lim_{k\to\infty} x_{n_k} = 0$, contrary to hypothesis. This contradiction establishes (4) and hence (3).

We are now in a position to prove that, regarded as an operator in W, T has its spectrum on the unit circumference. In fact, let ρ_W, σ_W denote the spectral radii of T and T^{-1} respectively, regarded as operators in W, and let $\|A\|_W$ denote the bound of an operator A on the unit sphere of W. We have $\rho_W = \lim_{n\to\infty} \|T^n\|_W^{1/n}$, and, for $x \in W$, we have

$$\gamma\|x\| \leqslant \{q(x)\}^{1/\nu} \leqslant \|x\|.$$

Thus, for $n = 1, 2, \ldots$, and $x \in W$, we have

$$\gamma\|T^n x\| \leqslant \{q(T^n x)\}^{1/\nu} = \{q(x)\}^{1/\nu} \leqslant \|x\|,$$

from which $\|T^n\|_W \leqslant \gamma^{-1}$, $\rho_W \leqslant 1$. On the other hand, we also have

$$\gamma\|x\| \leqslant \{q(x)\}^{1/\nu} = \{q(T^n x)\}^{1/\nu} \leqslant \|T^n x\|,$$

which shows that $\rho_W \geqslant 1$; therefore $\rho_W = 1$. Similarly, $\sigma_W = 1$. It is now clear that, when T is regarded as an operator in W, its spectrum lies on the unit circumference.

It remains only to prove the final assertion in the theorem. We note first that, if $u \in V$ is a characteristic vector of T with real characteristic number λ such that $|\lambda| \neq 1$, then $q(u) = 0$. In fact,

$$q(u) = q(Tu) = |\lambda|^\nu q(u).$$

Suppose now that $\rho > 1$, and that (for some $n = 1, 2,...$) v satisfies the conditions

$$(T-\rho I)^n v = 0, \qquad (T-\rho I)^{n-1} v \neq 0.$$

We prove that in this case $n = 1$, and v is linearly dependent on u_1. Let $v' = (T-\rho I)^{n-1}v$. Then $(T-\rho I)v' = 0$ and $v' \neq 0$. Since $\sigma^{-1} \neq \rho$, we have $\psi_2(v') = 0$. Since, as we have noted above, $|\phi(v')| = \|Qv'\|$, it now follows from Lemma 4 that v' is linearly dependent on u_1; i.e. $v' = \kappa u_1$. Writing

$$v = \psi_1(v)u_1 + \psi_2(v)u_2 + w \quad (w \in W),$$

we have

$$\kappa u_1 = (T-\rho I)^{n-1}v = \left(\frac{1}{\sigma} - \rho\right)^{n-1} \psi_2(v)u_2 + (T-\rho I)^{n-1}w, \tag{7}$$

provided that $(n-1) > 0$. Since $\kappa \neq 0$, equation (7) is impossible; therefore $n = 1$ and ρ is a simple characteristic number.

Similar considerations may be applied to σ^{-1} if $\sigma > 1$. This completes the proof of the theorem.

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ON ROTATIONS IN HILBERT SPACE

By C. R. PUTNAM (*Purdue University*)

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1. LET Ω denote the metric space the points of which are infinite orthogonal (real, unitary) matrices O and on which the distance between any two points O_1 and O_2 is $|O_1-O_2|$. Here the symbol $|A|$ (also $\|A\|$) denotes the usual norm of a bounded operator on a (real) Hilbert space. The subset of Ω consisting of those orthogonal matrices O admitting the representation $O = e^S$, where S is a real, bounded, skew-symmetric matrix, will be denoted by Ω_0. A point O of Ω will be called a *rotation* or a *reflection* according as O lies in Ω_0 or in $\Omega-\Omega_0$ [cf. (**1**), 591–2; for the terminology of this paper see (**2**) and (**3**)].

Various connectedness properties of the above-mentioned, as well as other, subsets of Ω have been obtained in (**2**) and (**3**). The proofs involved certain results on orthogonal matrices which were obtained in (**1**) and which were based on the spectral resolution of an arbitrary unitary matrix [(**4**) 268–77]. Furthermore, characterizations of the sets Ω_0 and $\Omega-\Omega_0$ in terms of the spectra of the matrices contained in these sets were obtained in (**3**). It was shown there that O is in Ω_0 if and only if -1 occurs in the point spectrum of O with an infinite or with a finite but even multiplicity (possibly zero).

Many topological questions relating to Ω_0 and $\Omega-\Omega_0$ remain open. For instance, it is not known whether these two sets are topologically equivalent, as are the corresponding sets in the case of a finite number of dimensions [cf. (**3**) 55]. Also, corresponding to the fact that, in the finite n-dimensional case ($n > 1$), the set of rotations is doubly connected, nothing seems to be known about the connectivity of Ω_0. It will be shown, however, in §§ 2–4 below, that *the set* Ω_0 *is locally arcwise connected*: that is, for every point A of Ω_0 and for every neighbourhood of A in Ω,

$$\mathfrak{N} = \mathfrak{N}_A\colon\ |O-A| < \epsilon \quad (\epsilon \text{ sufficiently small}),$$

there exists a neighbourhood

$$\mathfrak{M} = \mathfrak{M}_A\colon\ |O-A| < \delta,$$

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contained in N, such that any two rotations in $\mathfrak{M}$ can be connected by a continuous path lying in $\mathfrak{N}$ and in Ω_0. (Incidentally, it will remain undecided whether the set of reflections $\Omega-\Omega_0$ is also locally connected.) The complications arising in the proof of the above assertion stem from the fact that, unlike the situation in the corresponding finite-dimensional case, the space Ω is arcwise connected and the sets Ω_0 and $\Omega-\Omega_0$ are neither open nor closed in the space Ω [see (**2**) and (**3**)]. Moreover, the intersection of the closures of Ω_0 and $\Omega-\Omega_0$ is non-vacuous and, in fact, consists of those rotations and reflections for which -1 is in the essential spectrum [(**3**) 58]. (A number λ is said to be *in the essential spectrum* of O if it is either a cluster point of the spectrum of O or is in the point spectrum of O with an infinite multiplicity [cf. (**3**) 59].)

2. Let A denote any orthogonal matrix with an ϵ-neighbourhood

$$\mathfrak{N}_A\colon |O-A| < \epsilon \quad (< 2).$$

It will first be shown that any two points of $\mathfrak{N}_A$ can be joined in Ω by a path lying entirely in $\mathfrak{N}_A$. The transformation $O \to OA^{-1}$, where O is in $\mathfrak{N}_A$, is a homeomorphism of $\mathfrak{N}_A$ onto an ϵ-neighbourhood, say $\mathfrak{N}'$, of the unit matrix I (of Ω). Clearly, it is enough to show that any point O of $\mathfrak{N}'$ can be connected to I by a path in $\mathfrak{N}'$. Since $\epsilon < 2$, the point -1 is not in the spectrum of any point O of $\mathfrak{N}'$; hence $+1$ is not in the spectrum (in particular, not in the point spectrum) of $-O$. In terms of the spectral resolution of $-O$, namely

$$-O = \int_{+0}^{2\pi-0} e^{i\lambda}\, dE(\lambda),$$

define for every t on $0 \leqslant t \leqslant 1$ a matrix O_t by

$$O_t = \int_{+0}^{2\pi-0} e^{it(\lambda-\pi)}\, dE(\lambda).$$

Here the $E(\lambda)$ satisfy the conditions obtained in (**1**) [see 598–9] for an orthogonal matrix, and O_t for $0 \leqslant t \leqslant 1$ is a continuous path in Ω joining I $(= O_0)$ to the given O $(= O_1)$ [see (**3**) 71]. It is to be noted that the normalization is such that the $E(\lambda)$ are continuous from the left [cf. (**3**) 60, for a fuller explanation of terminology]. It is clear that

$$|I-O| = \max|1+e^{i\lambda}| = \max|e^{i(\lambda-\pi)}-1|$$

and that
$$|I-O_t| = \max|e^{it(\lambda-\pi)}-1|,$$

where, in each case, only those values on $0 \leqslant \lambda \leqslant 2\pi$ are considered for which $e^{i\lambda}$ is in the spectrum of $-O$ [cf. (**3**) 66]. It follows that

$$|I-O_t| \leqslant |I-O| \quad (0 \leqslant t \leqslant 1),$$

and so the path O_t lies in $\mathfrak{N}'$. The assertion at the beginning of this section now follows.

3. Next, let A be a point of Ω_0 which does not have -1 in its essential spectrum, so that A is an 'isolated rotation' [cf. (**3**) 58]. Then every neighbourhood of A in Ω,

$$\mathfrak{N}_A\colon |O-A| < \epsilon \quad (\epsilon \text{ sufficiently small})$$

consists only of points O belonging to Ω_0. The result of the preceding section then implies that any two points of $\mathfrak{N}_A$ can be joined by a path lying in $\mathfrak{N}_A$ (and hence also in Ω_0).

Next, let A be a point of Ω_0 which has -1 in its essential spectrum and consider any neighbourhood of A in Ω,

$$\mathfrak{N}_A\colon |O-A| < \epsilon.$$

It will be shown that any two rotations of $\mathfrak{N}_A$ can be connected by a path lying in a $(C\epsilon)$-neighbourhood of A ('sphere' of radius $C\epsilon$) and also in Ω_0, where C denotes a positive constant independent of ϵ and of the particular pair of points in $\mathfrak{N}_A$ considered, thus completing the proof of the italicized assertion of § 1. In order to show this, let R be any rotation of $\mathfrak{N}_A$. Then, as was shown in § 2, there exists a continuous path, say

$$O_s = \int\limits_0^{2\pi} e^{i\lambda}\, dE_s(\lambda) \quad (0 \leqslant s \leqslant 1) \tag{1}$$

in Ω joining A $(= O_0)$ to R $(= O_1)$ and lying in $\mathfrak{N}_A$. It will be shown that the path O_s can be modified so as to produce a path joining A to R and so as to lie also in Ω_0 and in a $(C\epsilon)$-neighbourhood of A.

To this end, let a function $t(s,\lambda)$ be defined for $0 \leqslant s \leqslant 1$ and $0 \leqslant \lambda \leqslant 2\pi$ as follows. If $s = 0$ or $s = 1$, let $t = t(s,\lambda) = 1$ for $0 \leqslant \lambda \leqslant 2\pi$. Let $h = h(s)$ be a continuous function on $0 \leqslant s \leqslant 1$ satisfying $0 < h(s) < \frac{1}{2}\pi$ for $0 < s < 1$ and $h(0) = h(1) = 0$. Let $t(s,\lambda)$ be the continuous function on $0 \leqslant \lambda \leqslant 2\pi$ (s fixed) whose graph consists of the five segments joining the successive points, in the (λ, t)-plane: $(0,1)$, $(\pi-2h, 1)$, $(\pi-h, 0)$, $(\pi+h, 0)$, $(\pi+2h, 1)$, and $(2\pi, 1)$. It is clear that $t(s,\lambda) = t(s, 2\pi-\lambda)$ for $0 \leqslant s \leqslant 1$ and $0 \leqslant \lambda \leqslant 2\pi$. Define for every s on $0 \leqslant s \leqslant 1$ a matrix

$$T_s = -\int\limits_0^{2\pi} e^{i(\lambda-\pi)t(s,\lambda)}\, dE_s(\lambda). \tag{2}$$

As a consequence of the definition of $t(s,\lambda)$ and the properties of the projections $E_s(\lambda)$ [cf. (**3**), 60 and 71], the locus T_s $(0 \leqslant s \leqslant 1)$ is a

continuous path lying in Ω and joining A $(= T_0)$ to R $(= T_1)$. Note that

$$f(s,\lambda) = e^{i(\lambda-\pi)t(s,\lambda)}$$

is a continuous function of the two variables s and λ on $0 \leqslant s \leqslant 1$, $0 \leqslant \lambda \leqslant 2\pi$, and satisfies $f(s,0) = f(s,2\pi)$ $(= -1)$. Clearly, the continuity of O_s implies that of T_s.

4. Two cases will be considered. Suppose first that -1 is in the essential spectrum of R (as well as of A). Let $k(s) = \min|\pi-\lambda(s)|$, where $e^{i\lambda(s)}$ is in the essential spectrum of O_s. As a consequence of the definition of the essential spectrum given at the end of § 1, the continuity of O_s implies that of the function $k(s)$ also. In view of the equalities $k(0) = k(1) = 0$ and the fact that O_s lies in the ϵ-neighbourhood $\mathfrak{N}_A$, one can define the above-mentioned $h(s)$ so that $h(s) > k(s)$ for $0 < s < 1$ and so that $|e^{ih}-1| < \epsilon$ for $0 \leqslant s \leqslant 1$ [cf. (**3**) 60–1 and 66]. Since $t(s,\lambda) \equiv 0$ for $0 < s < 1$ and $\pi-h(s) \leqslant \lambda \leqslant \pi+h(s)$, it is clear that -1 is in the point spectrum of T_s $(0 < s < 1)$ with an infinite multiplicity. Thus, every T_s $(0 \leqslant s \leqslant 1)$ is in Ω_0. Moreover, it is clear that

$$|T_s - O_s| < K\epsilon \quad (0 \leqslant s \leqslant 1), \tag{3}$$

where K denotes a constant independent of ϵ and of s. Consequently A can be joined to R by a path lying in Ω_0 and in some $(C\epsilon)$-neighbourhood $(C = K+1)$ of A.

The second case to be considered is that in which R does not have -1 in its essential spectrum. Hence, R is an isolated rotation; thus, all O of Ω sufficiently close to R lie in Ω_0. Let $s^* < 1$ be chosen so that all elements O_s of the path (1) for which $s^* \leqslant s \leqslant 1$ are isolated rotations. Then define $h = h(s)$ as above for $0 \leqslant s \leqslant s^*$, so that every T_s of (2) on this s-domain lies in Ω_0. On $s^* \leqslant s \leqslant 1$, it is only necessary to define $h(s)$ so as to be positive on $0 < s < 1$, be continuous and satisfy $|e^{ih}-1| < \epsilon$ on the entire interval $0 \leqslant s \leqslant 1$, and satisfy $h(1) = 0$. As long as $h(s) > k(s)$, it is clear that -1 is an eigenvalue of T_s with an infinite multiplicity, and so T_s is a rotation. On the other hand, $k(s)$ has a positive lower bound on $s^* \leqslant s \leqslant 1$ and for s sufficiently close to 1, $h(s) \leqslant k(s)$ (since h is continuous and $h(1) = 0$). Nevertheless, even for these values of s, T_s is still a rotation.

In order to see this, note that, if $s^* \leqslant s \leqslant 1$, then -1 occurs in the point spectrum of O_s with a (finite) even multiplicity, say m [cf. (**3**) 58]. Moreover, if $e^{i\lambda}$ is in the spectrum of O_s, so also is $e^{-i\lambda}$ [cf. (**1**) 698–700]. Consequently, if $h(s) \leqslant k(s)$, there exist either an infinity or a finite even number, say n, of eigenvalues $e^{i\lambda}$ of O_s whenever $\pi-h \leqslant \lambda \leqslant \pi+h$ (and

no points $e^{i\lambda}$ of the essential spectrum for $\pi-h < \lambda < \pi+h$). To each of these eigenvalues, there corresponds an eigenvalue -1 in the spectrum of T_s. (In fact, if $x \neq 0$ is any element of the Hilbert space, it is clear that

$$|(T_s+I)x|^2 = \int\limits_0^{2\pi} |1-e^{i(\lambda-\pi)t(s,\lambda)}|^2 \, d|E_s(\lambda)x|^2$$

[cf. (3) 71–2]. Hence, x is an eigenfunction of T_s belonging to the eigenvalue -1 if and only if x is in the range of the projection

$$E_s(\pi+h)-E_s(\pi-h).)$$

Thus, whenever $h(s) \leqslant k(s)$, T_s has -1 in its spectrum with a multiplicity which is either ∞ or $m+n$, and hence even, and consequently is a rotation. Thus the entire path T_s $(0 \leqslant s \leqslant 1)$ is in Ω_0 and, as before, satisfies (3). Hence T_s also lies in a $(C\epsilon)$-neighbourhood of A, and the proof of the italicized assertion of § 1 is now complete.†

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A SIMPLE PROOF OF SOME PARTITION FORMULAE OF RAMANUJAN'S

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1. Introduction

Bailey (2) showed that two of Ramanujan's formulae†

$$\sum_{0}^{\infty} p(5n+4)x^n = 5\prod_{1}^{\infty} \frac{(1-x^{5n})^5}{(1-x^n)^6} \tag{1.1}$$

and

$$\prod_{1}^{\infty} \frac{(1-x^n)^5}{1-x^{5n}} = 1-5\sum_{1}^{\infty} \left(\frac{n}{5}\right) \frac{nx^n}{1-x^n} \tag{1.2}$$

can be obtained from a well-known formula in elliptic functions. Formula (1.1) is deduced by Bailey in an earlier paper (1) from

$$x\prod_{1}^{\infty} \frac{(1-x^{5n})^5}{1-x^n} = \sum_{-\infty}^{\infty} \left[\frac{x^{5n+1}}{(1-x^{5n+1})^2} - \frac{x^{5n+2}}{(1-x^{5n+2})^2}\right] \tag{1.3}$$

by an argument of Ramanujan's (4), while (1.2) and (1.3) are special cases of

$$\sum_{-\infty}^{\infty} \left[\frac{xq^n}{(1-xq^n)^2} - \frac{yq^n}{(1-yq^n)^2}\right] = \frac{(x-y)(1-xy)}{(1-x)^2(1-y)^2} \times$$

$$\times \prod_{1}^{\infty} \frac{(1-xyq^n)(1-x^{-1}y^{-1}q^n)(1-xy^{-1}q^n)(1-x^{-1}yq^n)(1-q^n)^4}{(1-xq^n)^2(1-x^{-1}q^n)^2(1-yq^n)^2(1-y^{-1}q^n)^2}. \tag{1.4}$$

Formula (1.4) is obtained from the well-known formula

$$\wp(u)-\wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)} \tag{1.5}$$

when the functions are replaced by their Fourier series. I show how (1.4) can be obtained by a very simple method which does not require the use of an identity from another source.

† In (1.2) the symbol $\left(\frac{n}{5}\right)$ is the Legendre–Jacobi symbol.

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Carlitz (**3**) treats the formula

$$\wp'(u) = -\frac{\sigma(2u)}{\sigma^4(u)} \tag{1.6}$$

in a similar way to derive the formula†

$$\frac{1+x}{(1-x)^3}+x^{-1}\sum_1^\infty \frac{n^2q^n}{1-q^n}(x^n-x^{-n})$$
$$= \frac{(1+x)}{(1-x)^3}\prod_1^\infty \frac{(1-x^2q^n)(1-x^{-2}q^n)(1-q^n)^6}{(1-xq^n)^4(1-x^{-1}q^n)^4}. \tag{1.7}$$

Although (1.7) can be derived from (1.4) by dividing by $x-y$ and then letting $y \to x$, as Carlitz observes, we begin with a derivation of (1.7) to illustrate the method.

2. Proof of Carlitz's formula

Suppose that $0 < |q| < 1$ and denote by $f(x)$ the second member of (1.7). Then

$$f(x) = qf(qx) = q^{-1}f(q^{-1}x). \tag{2.1}$$

Now $f(x)$ has poles of order three at the points

$$x = q^k \quad (k = 0, \pm 1, \pm 2, \ldots).$$

By direct computation the principal part of $f(x)$ at $x = 1$ is found to be

$$\frac{2}{(1-x)^3}-\frac{1}{(1-x)^2}+\frac{0}{1-x} = \frac{1+x}{(1-x)^3}.$$

From (2.1) the principal part of $f(x)$ at $x = q^{-k}$ $(k = 1, 2, \ldots)$ is

$$\frac{q^k(1+xq^k)}{(1-xq^k)^3},$$

and at $x = q^{+k}$ $(k = 1, 2, \ldots)$ is

$$\frac{q^{-k}(1+xq^{-k})}{(1-xq^{-k})^3} = -x^{-2}\frac{q^k(1+x^{-1}q^k)}{(1-x^{-1}q^k)^3}.$$

Subtract the sum of the principal parts at the poles from $f(x)$ and define the remainder as $h(x)$: that is,

$$f(x) = h(x)+\frac{1+x}{(1-x)^3}+\sum_1^\infty \frac{q^k(1+xq^k)}{(1-xq^k)^3}-x^{-2}\sum_1^\infty \frac{q^k(1+x^{-1}q^k)}{(1-x^{-1}q^k)^3}. \tag{2.2}$$

† I have replaced q^2 by q and multiplied by $(1+x)/(1-x)^3$ for convenience.

From (2.1) we have $$h(x) = qh(qx). \tag{2.3}$$

Also, $h(x)$ is analytic everywhere, except possibly at 0 and ∞. Hence it has a (unique) Laurent expansion. Substituting in (2.3) and equating coefficients we find that $h(x)$ reduces to the form a/x. But it is evident that $a = 0$ by putting $x = -1$ in (2.2). Therefore $h(x)$ is identically zero.

Formula (2.2) with $h(x) \equiv 0$ is equivalent to (1.7), which becomes evident from the simple formula

$$\sum_{k=1}^{\infty} \frac{q^k(1+xq^k)}{(1-xq^k)^3} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^2 q^{kn} x^{n-1} = \sum_{n=1}^{\infty} \frac{n^2 q^n}{1-q^n} x^{n-1}.$$

3. Proof of Bailey's formula

Let $F(x,y)$ denote the second member of (1.4). To apply the method of § 2 we first treat y as a parameter (not equal to an integral power of q). Then $F(x,y)$, as a function of x, has poles of order two at

$$x = q^n \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and $$F(x,y) = F(qx,y) = F(q^{-1}x,y). \tag{3.1}$$

The principal part at $x = 1$ is

$$\frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

The fact that this expression is independent of y suggests that $F(x,y)$ can be written as $g(x)+h(y)$; I find it shorter to refuse this gambit.

Using (3.1) and the principal part at $x = 1$ we can write

$$\left.\begin{aligned} F(x,y) &= H(x,y)+G(x) \\ G(x) &= \frac{x}{(1-x)^2} + \sum_{1}^{\infty} \left[\frac{xq^k}{(1-xq^k)^2} + \frac{x^{-1}q^k}{(1-x^{-1}q^k)^2}\right] \end{aligned}\right\}, \tag{3.2}$$

where $H(x,y)$ has a Laurent expansion

$$H(x,y) = \sum_{-\infty}^{\infty} a_n(y) x^n, \qquad 0 < |x| < \infty$$

and satisfies the equation

$$H(x,y) = H(qx,y). \tag{3.3}$$

On using (3.3), $H(x,y)$ reduces to $a_0(y)$, which is easily found to be $-G(y)$ from (3.2) and the obvious relation

$$F(x,y) = -F(y,x).$$

This completes the proof of (1.4).

4. Remarks

The proof of (1.4) given in § 3, together with the elementary arguments given by Bailey (**1**, **2**) to deduce (1.2) and (1.3) from (1.4) and by Ramanujan (**4**) to deduce (1.1) from (1.3), constitutes the simplest known proof of Ramanujan's famous identities. This proof is both elementary and self-contained.

The method used in § 3 to prove (1.4) has been used by many investigators since Jacobi. It was used often by Cauchy. Although elementary, the method seems to be very useful in deriving additive-type identities, especially partition identities.

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ON THE PARSEVAL FORMULA IN THE THEORY OF EIGENFUNCTION EXPANSIONS ARISING FROM DIFFERENTIAL EQUATIONS

By A. I. MARTIN *and* E. C. TITCHMARSH (*Oxford*)

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1. CONSIDER a differential equation

$$\nabla^2\psi(\mathbf{x})+\{\lambda-q(\mathbf{x})\}\psi(\mathbf{x}) = 0, \tag{1.1}$$

where $\mathbf{x}$ stands for a finite number of variables x, y,..., each taking all real values, and

$$\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial y^2+\dots.$$

Let $G(\mathbf{x}, \boldsymbol{\xi}, \lambda)$ be the Green's function of the problem, and let

$$H(\mathbf{x}, \boldsymbol{\xi}, \mu) = \lim_{\delta\to 0}\int\limits_0^\mu \operatorname{im} G(\mathbf{x}, \boldsymbol{\xi}, u+i\delta)\, du.$$

Let $f(\mathbf{x})$ be an arbitrary real function of L^2 (integrable square) over the whole space, and let

$$f_\mu(\mathbf{x}) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} \{H(\mathbf{x}, \boldsymbol{\xi}, \mu)-H(\mathbf{x}, \boldsymbol{\xi}, -\mu)\}f(\boldsymbol{\xi})\, \mathbf{d\xi}. \tag{1.2}$$

Then the expansion formula related to the problem is

$$f(\mathbf{x}) = \lim_{\mu\to\infty} f_\mu(\mathbf{x}). \tag{1.3}$$

This was proved by Titchmarsh (**3**) in the two-dimensional case. If the Green's function is unique, the conditions assumed are that f and $(q-\nabla^2)f$ are L^2 over the whole plane. If the Green's function is not unique, the additional restriction is to be imposed that

$$f\frac{\partial G}{\partial r}-G\frac{\partial f}{\partial r} = o\left(\frac{1}{r}\right)$$

as $r^2 = x^2+y^2 \to \infty$. The result, under similar conditions, was extended by Martin (**1**) to higher dimensions.

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To obtain the corresponding Parseval formula, multiply (1.3) by a function $g(\mathbf{x})$ of L^2 and integrate. We obtain formally

$$\int_{-\infty}^{\infty} fg\,\mathbf{dx} = \lim_{\mu\to\infty}\int_{-\infty}^{\infty} f_\mu g\,\mathbf{dx}$$

$$= \lim_{\mu\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{H(\mathbf{x},\boldsymbol{\xi},\mu)-H(\mathbf{x},\boldsymbol{\xi},-\mu)\}f(\boldsymbol{\xi})g(\mathbf{x})\,\mathbf{dx}\mathbf{d\xi}$$

$$= \lim_{\mu\to\infty}\{J(f,g,\mu)-J(f,g,-\mu)\}, \tag{1.4}$$

where $$J(f,g,\mu) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\mathbf{x},\boldsymbol{\xi},\mu)f(\mathbf{x})g(\boldsymbol{\xi})\,\mathbf{dx}\mathbf{d\xi}.$$

In particular, if $g = f$, the formula is

$$\int_{-\infty}^{\infty} f^2\,\mathbf{dx} = J(f,\infty)-J(f,-\infty), \tag{1.5}$$

where $J(f,\mu) = J(f,f,\mu)$.

In the case of a discrete spectrum, $J(f,\mu)$ is a step-function. If f has the Fourier coefficient $c_n = \int f\psi_n\,\mathbf{dx}$, where ψ_n is the eigenfunction corresponding to the eigenvalue λ_n, then $J(f,\mu)$ has the *saltus* c_n^2 at λ_n (or a sum of such terms in the case of 'degeneracy'). Hence (1.5) becomes

$$\int_{-\infty}^{\infty} f^2\,\mathbf{dx} = \sum_{n=0}^{\infty} c_n^2, \tag{1.6}$$

the ordinary Parseval formula.

We shall now investigate the conditions under which these formulae hold.

2. Let G_R, H_R denote the Green's function, etc., for the problem for a finite region, e.g. $-R \leqslant x \leqslant R,\ldots$, with the eigenfunctions vanishing on the boundary of the region. Let $\lambda_{n,R}$ denote the eigenvalues in this problem, and $c_{n,R}$, $d_{n,R}$ the 'Fourier coefficients' of two functions f and g of L^2. Then the Parseval theorem for the finite region gives†

$$\int_R\int_R G_R(\mathbf{x},\boldsymbol{\xi},\lambda)f(\mathbf{x})g(\boldsymbol{\xi})\,\mathbf{dx}\mathbf{d\xi} = \sum_{n=0}^{\infty}\frac{c_{n,R}\,d_{n,R}}{\lambda_{n,R}-\lambda}.$$

† For the case of two dimensions, see Titchmarsh (**3**), § 10. The proof in three dimensions is similar. In the case of more than three dimensions, treated by Martin, we appear to require that either f or g is bounded. This condition can be removed when we arrive at (2.2) since H is L^2.

Hence, if α and β are real, and $\nu > 0$,

$$\int_R \int_R \int_\alpha^\beta \operatorname{im} G_R(\mathbf{x}, \boldsymbol{\xi}, \mu+i\nu) f(\mathbf{x}) g(\boldsymbol{\xi})\, \mathbf{dx}\mathbf{d\xi} d\mu$$

$$= \sum_{n=0}^{\infty} c_{n,R}\, d_{n,R} \int_\alpha^\beta \frac{\nu\, d\mu}{(\lambda_{n,R}-\mu)^2+\nu^2}. \quad (2.1)$$

The integral on the right-hand side does not exceed π. Hence, by using Schwarz's inequality for sums, together with the Bessel inequality appropriate to the finite region, we see that the absolute value of the right-hand side does not exceed

$$\pi\Big(\sum_{n=0}^{\infty} c_{n,R}^2 \sum_{n=0}^{\infty} d_{n,R}^2\Big)^{\frac{1}{2}} \leqslant \pi\Big(\int_R f^2\, \mathbf{dx} \int_R g^2\, \mathbf{dx}\Big)^{\frac{1}{2}} \leqslant \pi\Big(\int_{-\infty}^{\infty} f^2\, \mathbf{dx} \int_{-\infty}^{\infty} g^2\, \mathbf{dx}\Big)^{\frac{1}{2}}.$$

Assuming temporarily that f and g vanish outside finite regions R' and R'' respectively, we obtain on making $R \to \infty$, $\nu \to 0$,

$$\Big|\int_{R'} \int_{R''} \{H(\mathbf{x}, \boldsymbol{\xi}, \beta) - H(\mathbf{x}, \boldsymbol{\xi}, \alpha)\} f(\mathbf{x}) g(\boldsymbol{\xi})\, \mathbf{dx}\mathbf{d\xi}\Big|$$

$$\leqslant \pi\Big(\int_{-\infty}^{\infty} f^2\, \mathbf{dx} \int_{-\infty}^{\infty} g^2\, \mathbf{dx}\Big)^{\frac{1}{2}}. \quad (2.2)$$

Now, for any real μ,

$$\int_{-\infty}^{\infty} H^2(\mathbf{x}, \boldsymbol{\xi}, \mu)\, d\xi < K(x, |\mu|).$$

In the two-dimensional case, this follows from Lemma ξ of Titchmarsh (3), by making $v \to 0$ and using Fatou's theorem; and a similar argument is valid generally. Hence, if f is L^2 over the whole space, the integral

$$F(\mathbf{x}, \mu) = \int_{-\infty}^{\infty} H(\mathbf{x}, \boldsymbol{\xi}, \mu) f(\boldsymbol{\xi})\, \mathbf{d\xi}$$

is convergent for all $\mathbf{x}$ and μ. Hence, making $R' \to \infty$ in (2.2), we obtain

$$\Big|\int_{R''} \{F(\boldsymbol{\xi}, \beta) - F(\boldsymbol{\xi}, \alpha)\} g(\boldsymbol{\xi})\, \mathbf{d\xi}\Big| \leqslant \pi\Big(\int_{-\infty}^{\infty} f^2\, \mathbf{dx} \int_{-\infty}^{\infty} g^2\, \mathbf{dx}\Big)^{\frac{1}{2}}.$$

Let $g(\boldsymbol{\xi}) = F(\boldsymbol{\xi}, \beta) - F(\boldsymbol{\xi}, \alpha)$ in R'', and zero elsewhere. Then

$$\int_{R''} \{F(\boldsymbol{\xi}, \beta) - F(\boldsymbol{\xi}, \alpha)\}^2\, \mathbf{d\xi} \leqslant \pi^2 \int_{-\infty}^{\infty} f^2\, \mathbf{dx},$$

or, since R'' is arbitrary,

$$\int_{-\infty}^{\infty} \{F(\boldsymbol{\xi}, \beta) - F(\boldsymbol{\xi}, \alpha)\}^2\, \mathbf{d\xi} \leqslant \pi^2 \int_{-\infty}^{\infty} f^2\, \mathbf{dx}. \quad (2.3)$$

In particular, taking $\alpha = 0$, $F(\boldsymbol{\xi}, \alpha) = 0$, we see that $F(\boldsymbol{\xi}, \beta)$ is L^2 for every β. Hence $J(f, g, \mu)$ exists, as a repeated integral, for every μ, and

$$J(f, g, \mu) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(\mathbf{x}, \mu) g(\mathbf{x}) \, \mathbf{dx}. \tag{2.4}$$

Also

$$|J(f, g, \beta) - J(f, g, \alpha)| = \frac{1}{\pi} \left| \int_{-\infty}^{\infty} \{F(\mathbf{x}, \beta) - F(\mathbf{x}, \alpha)\} g(\mathbf{x}) \, \mathbf{dx} \right|$$

$$\leqslant \left(\int_{-\infty}^{\infty} f^2 \, \mathbf{dx} \int_{-\infty}^{\infty} g^2 \, \mathbf{dx} \right)^{\frac{1}{2}}, \tag{2.5}$$

by (2.3).

On taking $g = f$, $d_{n,R} = c_{n,R}$ in (2.1), we obtain

$$\int_R \int_R \int_\alpha^\beta \operatorname{im} G_R . f(\mathbf{x}) f(\boldsymbol{\xi}) \, \mathbf{dx} \mathbf{d}\boldsymbol{\xi} d\mu \geqslant 0.$$

Hence, as before,

$$\int_{R'} \int_{R'} \{H(\mathbf{x}, \boldsymbol{\xi}, \beta) - H(\mathbf{x}, \boldsymbol{\xi}, \alpha)\} f(\mathbf{x}) f(\boldsymbol{\xi}) \, \mathbf{dx} \mathbf{d}\boldsymbol{\xi} \geqslant 0.$$

Making $R' \to \infty$, we see that, for any f of L^2,

$$J(f, \beta) - J(f, \alpha) \geqslant 0, \tag{2.6}$$

i.e. $J(f, \mu)$ is a non-decreasing function of μ. Also

$$J(f, \beta) - J(f, \alpha) \leqslant \int_{-\infty}^{\infty} f^2 \, \mathbf{dx},$$

as a particular case of (2.5). Thus

$$J(f, \infty) - J(f, -\infty) \leqslant \int_{-\infty}^{\infty} f^2 \, \mathbf{dx}. \tag{2.7}$$

This is the generalization of the Bessel inequality.

3. If f satisfies the conditions assumed for the validity of (1.3), then $f_\mu(\mathbf{x})$ is bounded for all real μ, and $\mathbf{x}$ in a bounded region; e.g. in the case of two dimensions this follows from Lemma ρ of Titchmarsh (**3**), on making $\operatorname{im} \lambda \to 0$. Hence, if g is any function of L^2,

$$\lim_{\mu \to \infty} \int_R f_\mu g \, \mathbf{dx} = \int_R f g \, \mathbf{dx}. \tag{3.1}$$

Now, on taking $\beta = -\alpha = \mu$ in (2.3), we obtain, for any f of L^2,

$$\int_{-\infty}^{\infty} f_\mu^2 \, \mathbf{dx} \leqslant \int_{-\infty}^{\infty} f^2 \, \mathbf{dx}. \tag{3.2}$$

Hence, if CR denotes the set complementary to R,

$$\left|\int\limits_{CR} f_\mu g\,\mathbf{dx}\right| \leqslant \left(\int\limits_{CR} f_\mu^2\,\mathbf{dx} \int\limits_{CR} g^2\,\mathbf{dx}\right)^{\frac{1}{2}}$$

$$\leqslant \left(\int\limits_{-\infty}^{\infty} f^2\,\mathbf{dx} \int\limits_{CR} g^2\,\mathbf{dx}\right)^{\frac{1}{2}}, \tag{3.3}$$

which can be made arbitrarily small by choice of R. Hence, if f satisfies the special conditions, (3.1) and (3.3) give

$$\lim_{\mu\to\infty} \int\limits_{-\infty}^{\infty} f_\mu g\,\mathbf{dx} = \int\limits_{-\infty}^{\infty} fg\,\mathbf{dx}. \tag{3.4}$$

Since
$$J(f,\mu)-J(f,-\mu) = \int\limits_{-\infty}^{\infty} f_\mu f\,\mathbf{dx}, \tag{3.5}$$

the Parseval formula (1.5) holds in this case.

To prove it for any function f of L^2, let $f^{(k)}$ be a sequence of functions, each of which satisfies the special conditions, and such that $f^{(k)}$ converges in mean square to f over the whole space. Then we can choose k so large that

$$\int\limits_{-\infty}^{\infty} (f-f^{(k)})^2\,\mathbf{dx} \leqslant \epsilon^2. \tag{3.6}$$

Let g be any function of L^2. Then, by the Schwarz inequality,

$$\left|\int\limits_{-\infty}^{\infty} (f-f^{(k)})g\,\mathbf{dx}\right| \leqslant \epsilon \int\limits_{-\infty}^{\infty} g^2\,\mathbf{dx}.$$

By the Schwarz inequality, (3.2) applied to $f-f^{(k)}$, and (3.6),

$$\left|\int\limits_{-\infty}^{\infty} (f_\mu-f_\mu^{(k)})g\,\mathbf{dx}\right| \leqslant \epsilon \int\limits_{-\infty}^{\infty} g^2\,\mathbf{dx}.$$

By (3.4),
$$\lim_{\mu\to\infty} \int\limits_{-\infty}^{\infty} (f^{(k)}-f_\mu^{(k)})g\,\mathbf{dx} = 0.$$

Altogether
$$\limsup \left|\int\limits_{-\infty}^{\infty} (f-f_\mu)g\,\mathbf{dx}\right| \leqslant 2\epsilon \int\limits_{-\infty}^{\infty} g^2\,\mathbf{dx},$$

and so in fact the left-hand side is zero. This proves (3.4), and so also (1.4) and (1.5), for any f and g of L^2.

The above argument shows that f_μ converges weakly to f, if f is L^2. But, by (3.2),

$$\int_{-\infty}^{\infty} (f_\mu - f)^2\,\mathbf{dx} = \int_{-\infty}^{\infty} f_\mu^2\,\mathbf{dx} + \int_{-\infty}^{\infty} f^2\,\mathbf{dx} - 2\int_{-\infty}^{\infty} f_\mu f\,\mathbf{dx}$$

$$\leqslant 2\int_{-\infty}^{\infty} f^2\,\mathbf{dx} - 2\int_{-\infty}^{\infty} f_\mu f\,\mathbf{dx},$$

which tends to zero, by the weak convergence. Hence in fact f_μ converges to f in mean square over the whole space.

4. A related formula

Let
$$D(f,g) = \int_{-\infty}^{\infty} \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \dots + qfg\right)\mathbf{dx} \tag{4.1}$$

for any functions f and g for which the integral exists. If we can integrate by parts, and the terms at the limits vanish, we obtain

$$D(f,g) = \int_{-\infty}^{\infty} (-f\nabla^2 g + qfg)\,\mathbf{dx} = \int_{-\infty}^{\infty} fg^*\,\mathbf{dx}, \tag{4.2}$$

where $g^* = qg - \nabla^2 g$.

In the case of a discrete spectrum, with eigenvalues and eigenfunctions λ_n, ψ_n, if $g = \psi_n$, then $g^* = \lambda_n \psi_n$, so that

$$D(f,\psi_n) = \lambda_n \int_{-\infty}^{\infty} f\psi_n\,\mathbf{dx} = \lambda_n c_n. \tag{4.3}$$

Taking $f = \psi_n$, we obtain

$$D(\psi_m,\psi_n) = \lambda_n \quad (m = n), \qquad D(\psi_m,\psi_n) = 0 \quad (m \neq n). \tag{4.4}$$

Let $D(f) = D(f,f)$. Then, by (4.2) and the Parseval formula,

$$D(f) = \int_{-\infty}^{\infty} ff^*\,\mathbf{dx} = \sum_{n=0}^{\infty} c_n c_n^*,$$

where

$$c_n^* = \int_{-\infty}^{\infty} f^*\psi_n\,\mathbf{dx} = \int_{-\infty}^{\infty} f\psi_n^*\,\mathbf{dx} = \lambda_n \int_{-\infty}^{\infty} f\psi_n\,\mathbf{dx} = \lambda_n c_n.$$

Hence we obtain formally

$$D(f) = \sum_{n=0}^{\infty} \lambda_n c_n^2. \tag{4.5}$$

[For the case of a finite two-dimensional region, see Courant and Hilbert, *Methoden der math. Physik*, i, 2 Aufl., vi, § 7 (53).]

The generalization of (4.5) to the case of possibly continuous spectra will clearly be

$$D(f) = \int_{-\infty}^{\infty} \lambda \, d\, J(f,\lambda). \tag{4.6}$$

We next consider the conditions under which this formula holds.

Consider for example the two-dimensional case, and suppose that f satisfies the conditions, stated in § 1, for the validity of the expansion formula. Let

$$\Phi(\mathbf{x},\lambda) = \Phi(\mathbf{x},\lambda,f) = -\int_{-\infty}^{\infty} G(\mathbf{x},\boldsymbol{\xi},\lambda)f(\boldsymbol{\xi})\,\mathbf{d}\boldsymbol{\xi},$$

and $\Phi^*(\mathbf{x},\lambda) = \Phi(\mathbf{x},\lambda,f^*)$. Then, by Lemma π of Titchmarsh (**3**),

$$\Phi^*(\mathbf{x},\lambda) = \lambda\,\Phi(\mathbf{x},\lambda) - f(\mathbf{x}).$$

Put $\lambda = \mu + i\nu$, take imaginary parts, and integrate with respect to μ over $(0,u)$. This gives

$$\int_0^u \operatorname{im}\Phi^*\,d\mu = \int_0^u \mu \operatorname{im}\Phi\,d\mu + \nu\int_0^u \operatorname{re}\Phi\,d\mu.$$

Multiplying by $f(\mathbf{x})$ and integrating with respect to each variable over $(-X,X)$, we obtain

$$\int_{-X}^{X} f(\mathbf{x})\,\mathbf{dx}\int_0^u \operatorname{im}\Phi^*\,d\mu = \int_0^u \mu\,d\mu \int_{-X}^{X} f\operatorname{im}\Phi\,\mathbf{dx} + $$
$$+\nu\int_{-X}^{X} f(\mathbf{x})\,\mathbf{dx}\int_0^u \operatorname{re}\Phi\,d\mu. \tag{4.7}$$

Now let $\nu \to 0$. The left-hand side tends to

$$\int_{-X}^{X} f(\mathbf{x})\,F^*(\mathbf{x},u)\,\mathbf{dx},$$

where $F^*(x,u) = F(x,u,f^*)$. The process is justified since, by Lemma ξ of Titchmarsh (**3**), if f is L^2,

$$\int_0^u \operatorname{im}\Phi\,d\mu \to F(x,u)$$

boundedly over finite ranges of $\mathbf{x}$ and u. Similarly, on integrating the first term on the right by parts, we obtain

$$u\int_0^u d\mu \int_{-X}^{X} f\operatorname{im}\Phi\,\mathbf{dx} - \int_0^u dt\int_0^t d\mu\int_{-X}^{X} f\operatorname{im}\Phi\,\mathbf{dx}$$
$$\to u\int_{-X}^{X} f(\mathbf{x})\,F(\mathbf{x},u)\,\mathbf{dx} - \int_0^u dt\int_{-X}^{X} f(\mathbf{x})\,F(\mathbf{x},t)\,\mathbf{dx}.$$

The last term on the right of (4.7) tends to zero. Considering first the case of a finite region, we have

$$\int_R \left|\int_0^u \Phi_R\,d\mu\right|^2 \mathbf{dx} = \sum_{n=0}^{\infty} |c_{n,R}|^2 \left|\int_0^u \frac{d\mu}{\lambda_n-\mu-i\nu}\right|^2$$

$$\leqslant \sum_{n=0}^{\infty} |c_{n,R}|^2 \int_0^u d\mu \int_0^u \frac{d\mu}{(\lambda_n-\mu)^2+\nu^2}$$

$$\leqslant \frac{\pi u}{\nu} \sum_{n=0}^{\infty} |c_{n,R}|^2 = \frac{\pi u}{\nu} \int_R f^2\,\mathbf{dx}.$$

Making $R \to \infty$, we obtain

$$\int_{-\infty}^{\infty} \left|\int_0^u \Phi\,d\mu\right|^2 \mathbf{dx} \leqslant \frac{\pi u}{\nu} \int_{-\infty}^{\infty} f^2\,\mathbf{dx}.$$

Hence the absolute value of the last term on the right of (4.7) does not exceed

$$\nu\left(\int_{-X}^{X} f^2\,\mathbf{dx}\right)^{\frac{1}{2}}\left(\int_{-X}^{X}\left\{\int_0^u \mathrm{re}\,\Phi\,d\mu\right\}^2 \mathbf{dx}\right)^{\frac{1}{2}}$$

$$\leqslant \nu\left(\int_{-\infty}^{\infty} f^2\,\mathbf{dx}\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\left|\int_0^u \Phi\,d\mu\right|^2 \mathbf{dx}\right)^{\frac{1}{2}} \leqslant (\pi u\nu)^{\frac{1}{2}} \int_{-\infty}^{\infty} f^2\,\mathbf{dx},$$

which tends to zero with ν. It follows that

$$\int_{-X}^{X} f(\mathbf{x})\,F^*(\mathbf{x},u)\,\mathbf{dx} = u\int_{-X}^{X} f(\mathbf{x})\,F(\mathbf{x},u)\,\mathbf{dx} - \int_0^u dt \int_{-X}^{X} f(\mathbf{x})\,F(\mathbf{x},t)\,\mathbf{dx}.$$

Making $X \to \infty$, we obtain

$$J(f,f^*,u) = u\,J(f,u) - \int_0^u J(f,t)\,dt = \int_0^u \mu\,d\,J(f,\mu). \tag{4.8}$$

From this, and (1.4) with $g = f^*$, it follows that

$$\int_{-\infty}^{\infty} ff^*\,\mathbf{dx} = \int_{-\infty}^{\infty} \mu\,d\,J(f,\mu) \tag{4.9}$$

provided that f satisfies the conditions stated in § 1 for the expansion formula.

We can now show *that* (4.6) *holds if f satisfies the conditions assumed for the expansion formula, if $\partial f/\partial x$ and $\partial f/\partial y$ are L^2, and qf^2 is L.*

For by integration by parts

$$\int\limits_{r\leqslant R}\left(1-\frac{r}{R}\right)ff^*\,\mathbf{dx} = \int\limits_{r\leqslant R}\left(1-\frac{r}{R}\right)\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+qf^2\right\}\mathbf{dx}-$$

$$-\frac{1}{R}\int\limits_{r\leqslant R} f\left(\frac{x}{r}\frac{\partial f}{\partial x}+\frac{y}{r}\frac{\partial f}{\partial y}\right)\mathbf{dx},$$

and the second term on the right tends to zero as $R\to\infty$. The result therefore follows from (4.9).

5. In certain cases the inequality

$$\int\limits_{-\infty}^{\infty}\lambda\,dJ(f,\lambda)\leqslant D(f) \tag{5.1}$$

can be proved under wider conditions than those which we have assumed for (4.6). Suppose first that the region is finite, with boundary condition $\psi=0$, and that f, $\partial f/\partial x$, $\partial f/\partial y$ are L^2 over the region, and $f=0$ on the boundary. Then the formulae (4.3) and (4.4) hold. Let $q\geqslant 0$. Then

$$\begin{aligned}0&\leqslant D\Big(f-\sum_{n=0}^{N}c_n\psi_n\Big)\\&=D(f)+\sum_{n=0}^{N}\lambda_n c_n^2-2\sum_{n=0}^{N}\lambda_n c_n^2\\&=D(f)-\sum_{n=0}^{N}\lambda_n c_n^2.\end{aligned}$$

Making $N\to\infty$, we obtain

$$\sum_{n=0}^{\infty}\lambda_n c_n^2\leqslant D(f). \tag{5.2}$$

This is equivalent to (5.1) in the case considered. The result also holds for any bounded q since this reduces to the previous case by a change of λ-origin.

Suppose next that the region is infinite, q is bounded below, the spectrum is discrete, and that f, $\partial f/\partial x$, $\partial f/\partial y$, and $|q|^{\frac{1}{2}}f$ are L^2. Then (4.3) and (4.4) again hold [cf. Titchmarsh (**2**), § 3 and § 14]. Hence (5.2) follows again. Now consider the general case. By the Parseval theorem for a finite region,

$$\int\limits_{R}\Phi_R(\mathbf{x},\lambda)f(\mathbf{x})\,\mathbf{dx}=\sum_{n=0}^{\infty}\frac{c_{n,R}^2}{\lambda-\lambda_{n,R}}=\int\limits_{-\infty}^{\infty}\frac{dJ_R(f,t)}{\lambda-t}=-\int\limits_{-\infty}^{\infty}\frac{J_R(f,t)}{(\lambda-t)^2}\,dt.$$

The function $J_R(f,t)$ is monotonic in t, and bounded for all R and t, by the analysis of § 2. It therefore follows from the Helly selection theorem (see e.g. Widder, *The Laplace Transform*, 27) that there is a sequence of values of R through which $J_R(f,\lambda)$ tends to a limit, say $j(f,\lambda)$, uniformly over any finite λ-range. On making $R \to \infty$, we therefore obtain

$$\int_{-\infty}^{\infty} \Phi(\mathbf{x},\lambda)f(\mathbf{x})\,d\mathbf{x} = -\int_{-\infty}^{\infty} \frac{j(f,t)}{(\lambda-t)^2}\,dt = \int_{-\infty}^{\infty} \frac{dj(f,t)}{\lambda-t}.$$

The inverse formula is

$$j(f,\mu_2)-j(f,\mu_1) = \frac{1}{\pi}\lim_{\nu\to 0}\int_{\mu_1}^{\mu_2}\left\{\operatorname{im}\int_{-\infty}^{\infty} \Phi(\mathbf{x},\lambda)f(\mathbf{x})\,d\mathbf{x}\right\}d\mu.$$

Since this is also the formula for $J(f,t)$,

$$j(f,t) = J(f,t).$$

Now suppose that the spectrum, both for finite and infinite regions, is bounded below, at λ_0. Let f satisfy the above conditions and vanish outside a finite region. Then, for all sufficiently large R,

$$\int_{\lambda_0}^{\infty} \lambda\,d\,J_R(f,\lambda) \leqslant D_R(f)$$

by (5.2) applied to the finite region R. Hence, if $\Lambda > 0$,

$$\int_{\lambda_0}^{\Lambda} \lambda\,d\,J_R(f,\lambda) \leqslant D_R(f).$$

Making $R \to \infty$ through a suitable sequence, we obtain

$$\int_{\lambda_0}^{\Lambda} \lambda\,d\,J(f,\lambda) \leqslant D(f), \tag{5.3}$$

and, since Λ is arbitrary, (5.1) follows. This case is required for an application elsewhere. No doubt the result can be obtained for more general functions f.

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ON THE COVERING OF LATTICE POINTS BY CONVEX REGIONS (II)

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1. In a recent note† in this Journal I proved a result concerning the covering of points of the integral lattice in the plane by symmetrical convex regions. I have since observed that a modification of the method proves a considerably stronger result which may be stated in the following form.

Theorem. *Let K be a plane convex region which is such that, however it is displaced (a displacement consists of a translation and a rotation), it contains some point of the integral lattice. Then its area $A(K)$ satisfies*

$$A(K) \geqslant \tfrac{4}{3},$$

with strict inequality unless K is congruent to the region K^* given by

$$|x| \leqslant \tfrac{1}{2}, \qquad |y| \leqslant \tfrac{3}{4}-x^2.$$

This result proves a conjecture‡ of J. L. Massera and J. J. Schäffer, to which my attention has very recently been drawn by Dr. Schäffer, and forms a partial solution to a problem§ suggested by L. A. Santaló.

The notation of my previous paper will be used, and also some of the lemmas, which will be referred to by their original numbers; the lemmas of this note will be distinguished by letters.

2. Massera and Schäffer‡ remark that K contains a unit square. This is most easily seen by using a result|| due to Emch, which states that four points forming the vertices of a square exist on the boundary of any bounded convex region. Denote this square by $ABCD$ and suppose that its side is less than 1. Displace K so that the centre of $ABCD$ is at the point $(\frac{1}{2}, \frac{1}{2})$ and the sides of $ABCD$ are parallel to the coordinate axes. Then by convexity K is contained in the crossing pair of strips formed by the lines of which AB, BC, CD, DA are segments. Since these strips contain no lattice point, we have a contradiction, and so $ABCD$ has sides at least equal to 1. Let U be a fixed unit square in K and without

† (2) 4 (1953) 284–92.
‡ *Publ. del Inst. de Mat. y Est.*, Montevideo, II (1951) 55–74.
§ *Publ. del Inst. de Mat.*, Rosario, I, 54.
|| *American J. of Math.* 35 (1913) 407–12.

loss of generality suppose that its centre is O and its sides are parallel to the coordinate axes.

Let d denote the greatest radial distance from O to the boundary of K. Then by a simple modification of Lemma 2 we have

$$A(K) \geqslant \tfrac{3}{4}+\tfrac{1}{2}(d^2-\tfrac{1}{4})^{\frac{1}{2}},$$

from which $A(K) > \frac{4}{3}$ if

$$d > \tfrac{1}{6}\sqrt{58} = 1{\cdot}269\ldots.$$

We thus suppose $$d < 1{\cdot}27. \tag{a}$$

By Lemma 3, for each θ the set $K(0,0,\theta)+\Gamma$ contains the square S. But the four squares $U(0,0,\theta)$, $U(0,1,\theta)$, $U(1,0,\theta)$, $U(1,1,\theta)$ together cover all of S except for a square gap in the centre of S. It is easily seen that for any θ this gap lies inside the circle whose centre is $(\frac{1}{2},\frac{1}{2})$ and whose radius is $1-\frac{1}{2}\sqrt{2}$. Thus the distance from any lattice point other than a vertex of S to any point of the gap is at least

$$\tfrac{1}{2}\sqrt{10}-(1-\tfrac{1}{2}\sqrt{2}) = 1{\cdot}288\ldots.$$

Hence by (a) the square S is covered by the four regions $K(0,0,\theta)$, $K(0,1,\theta)$, $K(1,0,\theta)$, $K(1,1,\theta)$ alone.

3. The main lemmas are analogous to Lemmas 4 and 5. We use the properties of

$$\phi(u,v) \equiv u^2+v^2+2\alpha(v-u)-2\beta^2.$$

Lemma A. *Let k be a positive constant. If $f_1(u), f_2(u)$ are functions continuous in the interval $-\beta-k \leqslant u \leqslant \beta+k$ and such that*

$$f_1(u-k)+f_2(v+k) \geqslant 0, \tag{b}$$

$$f_1(v-k)+f_2(u+k) \geqslant 0, \tag{c}$$

whenever $$\phi(u,v) = 0, \qquad |u| < \beta, \qquad |v| < \beta, \tag{d}$$

then $$\int_{-\beta}^{\beta} \{f_1(u-k)+f_2(u+k)\}\,du \geqslant 0,$$

with strict inequality unless there is always equality in (b) *and* (c).

Proof. If v is defined as a function of u by (d), then

$$f_1(u-k)+f_2(v+k), \qquad f_1(v-k)+f_2(u+k)$$

are continuous functions of u on the interval $-\beta \leqslant u \leqslant \beta$. Using

(b) and (c) and recalling that $\alpha \geqslant \beta$, we have

$$\int_{-\beta}^{\beta} (\alpha-u)\{f_1(u-k)+f_2(v+k)\}\,du \geqslant 0, \tag{e}$$

$$\int_{-\beta}^{\beta} (\alpha-u)\{f_1(v-k)+f_2(u+k)\}\,du \geqslant 0, \tag{f}$$

which, by (4) and (5), we may rewrite as

$$\int_{-\beta}^{\beta} (\alpha-u)f_1(u-k)\,du + \int_{-\beta}^{\beta} (\alpha+v)f_2(v+k)\,dv \geqslant 0,$$

$$\int_{-\beta}^{\beta} (\alpha+v)f_1(v-k)\,dv + \int_{-\beta}^{\beta} (\alpha-u)f_2(u+k)\,du \geqslant 0.$$

On addition we have

$$2\alpha \int_{-\beta}^{\beta} \{f_1(u-k)+f_2(u+k)\}\,du \geqslant 0.$$

There is strict inequality here unless there is equality in both (e) and (f). But, since the integrands in (e) and (f) are continuous, there is equality in (e) and (f) only when there is always equality in (b) and (c).

Lemma B. *If $g_1(u)$, $g_2(u)$ are functions continuous in the interval $-\beta-k \leqslant u \leqslant \beta+k$, and such that, when u, v satisfy* (d),

$$g_1(u-k)+g_2(v+k) \geqslant v-u+2h,$$
$$g_1(v-k)+g_2(u+k) \geqslant v-u+2h,$$

where h is a constant, then

$$\int_{-\beta}^{\beta} \{g_1(u-k)+g_2(u+k)\}\,du \geqslant 4\beta h+\frac{4\beta^3}{3\alpha}.$$

Proof. We write

$$g_0(u) = h+(\beta^2-u^2)/2\alpha,$$

$$f_1(u-k) = g_1(u-k)-g_0(u), \qquad f_2(u+k) = g_2(u+k)-g_0(u).$$

Then

$$f_1(u-k)+f_2(v+k) \geqslant v-u+2h-\frac{2\beta^2-u^2-v^2}{2\alpha}-2h = 0,$$

$$f_1(v-k)+f_2(u+k) \geqslant v-u+2h-\frac{2\beta^2-v^2-u^2}{2\alpha}-2h = 0,$$

so that, by Lemma A,

$$\int_{-\beta}^{\beta} \{g_1(u-k)+g_2(u+k)\}\,du = 2\int_{-\beta}^{\beta} g_0(u)\,du + \int_{-\beta}^{\beta} \{f_1(u-k)+f_2(u+k)\}\,du$$
$$\geqslant 2\int_{-\beta}^{\beta} g_0(u)\,du$$
$$= 4\beta h+\tfrac{4}{3}\beta^3\alpha^{-1}.$$

4. Proof of the theorem

We follow closely the proof of the earlier theorem. The unit square $|x| \leqslant \frac{1}{2}$, $|y| \leqslant \frac{1}{2}$ is contained in K. We write

$$\inf_K x = -\tfrac{1}{2}-h-k, \qquad \sup_K x = \tfrac{1}{2}+h-k,$$
$$\inf_K y = -\tfrac{1}{2}-h'-k', \qquad \sup_K y = \tfrac{1}{2}+h'-k',$$

where $$h \geqslant |k| \geqslant 0, \qquad h' \geqslant |k'| \geqslant 0.$$

By remarking that K contains the convex closure of the unit square and the extreme points we have

$$A(K) \geqslant 1+h+h'.$$

There is no loss of generality in supposing that $h \leqslant h'$, $k \geqslant 0$, so that

$$A(K) \geqslant 1+2h,$$

and we are concerned only with the range of values

$$0 \leqslant k \leqslant h \leqslant \tfrac{1}{6}.$$

When $$\cos\theta+\sin\theta \geqslant 1+2h,$$

the strip $$\tfrac{1}{2}+h-k < x\cos\theta+y\sin\theta < \cos\theta+\sin\theta-\tfrac{1}{2}-h-k$$

contains no point of $K(0,0,\theta)$ or $K(1,1,\theta)$. Hence some point on each of the lines

$$x\cos\theta+y\sin\theta = \tfrac{1}{2}+h-k, \tag{g}$$
$$(x-1)\cos\theta+(y-1)\sin\theta = -\tfrac{1}{2}-h-k \tag{h}$$

belongs to both $K(0,1,\theta)$ and $K(1,0,\theta)$. But K is defined by equations of the form

$$y \leqslant \tfrac{1}{2}+g_1(x), \qquad -y \leqslant \tfrac{1}{2}+g_2(-x),$$

where g_1 and g_2 are continuous functions, satisfying

$$g_1(x) \geqslant 0, \qquad g_2(x) \geqslant 0 \tag{i}$$

when $|x| \leqslant \frac{1}{2}$. Hence all points of $K(1,0,\theta)$ satisfy

$$y\cos\theta-(x-1)\sin\theta \leqslant \tfrac{1}{2}+g_1\{(x-1)\cos\theta+y\sin\theta\},$$

while all points of $K(0,1,\theta)$ satisfy

$$(1-y)\cos\theta+x\sin\theta \leqslant \tfrac{1}{2}+g_2\{-x\cos\theta-(y-1)\sin\theta\}.$$

Using (g) we obtain on addition

$$\cos\theta+\sin\theta \leqslant 1+g_1(\tfrac{1}{2}+h-k-\cos\theta)+g_2(-\tfrac{1}{2}-h+k+\sin\theta),$$

and using (h) we obtain on addition

$$\cos\theta+\sin\theta \leqslant 1+g_1(-\tfrac{1}{2}-h-k+\sin\theta)+g_2(\tfrac{1}{2}+h+k-\cos\theta).$$

Writing $\qquad u = \tfrac{1}{2}+h-\cos\theta, \qquad v = -\tfrac{1}{2}-h+\sin\theta,$

we thus have

$$g_1(u-k)+g_2(v+k) \geqslant v-u+2h,$$
$$g_1(v-k)+g_2(u+k) \geqslant v-u+2h.$$

We note that

$$\beta^2 = \tfrac{1}{2}-(\tfrac{1}{2}+h)^2 = \tfrac{1}{4}-h-h^2 \leqslant (\tfrac{1}{2}-h)^2 \leqslant (\tfrac{1}{2}-k)^2,$$

and so $\qquad \beta+k \leqslant \tfrac{1}{2}.$

We now have, by (i), Lemmas 6, B,

$$\begin{aligned} A(K) &\geqslant 1+h+\int_{-\frac{1}{2}}^{\frac{1}{2}} \{g_1(x)+g_2(x)\}\,dx \\ &\geqslant 1+h+\int_{-\beta}^{\beta} \{g_1(x-k)+g_2(x+k)\}\,dx \\ &\geqslant 1+h+4\beta h+\tfrac{4}{3}\beta^3\alpha^{-1} \\ &\geqslant \tfrac{4}{3}. \end{aligned}$$

The last inequality is strict unless $h = 0$, which gives $k = 0$. It now only remains to show that there is strict inequality in the theorem except when

$$g_1(u) \equiv g_2(u) \equiv g_0(u),$$

i.e. except when $\qquad f_1(u) \equiv f_2(u) \equiv 0.$

Suppose for some u_0 that f_1, say, is non-zero, i.e.

$$f_1(u_0) = p \neq 0.$$

There is strict inequality in the theorem unless (b) and (c) are always true with equality. In this case, defining a sequence $\{u_n\}$ by the recursion

$$\phi(u_{n-1}, u_n) = 0$$

we have for all m

$$f_1(u_{2m}) = p, \qquad f_2(u_{2m+1}) = -p.$$

As $n \to \infty$, $u_n \to \beta = \frac{1}{2}$, so that by the continuity of f_1 and f_2 we have

$$f_1(\tfrac{1}{2}) = p, \qquad f_2(\tfrac{1}{2}) = -p.$$

Since $g_0(\frac{1}{2}) = 0$, we have

$$g_1(\tfrac{1}{2}) = p, \qquad g_2(\tfrac{1}{2}) = -p,$$

contradicting (i). This completes the proof.

[*Note added in proof, 18 May* 1955. The result of this paper has been discovered independently by Schäffer, whose work is in course of publication in *Mathematische Annalen*. I am grateful to Dr. Schäffer for pointiing out to me that the result attributed in § 2 to Emch has only recently been completely proved, by Christensen, *Mat. Tidsskrift* B (1950) 22–6.]

INTEGRAL TRANSFORMS AND EIGEN-FUNCTION THEORY (II)

By D. B. SEARS (*Cape Town*)

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1. LET $\phi(x,\lambda)$ be the solution of the differential equation

$$\frac{d^2y}{dx^2}+\{\lambda-q(x)\}y=0 \quad (x\geqslant 0)$$

satisfying the boundary conditions

$$\phi(0,\lambda)=\sin\alpha, \qquad \phi'(0,\lambda)=-\cos\alpha,$$

primes denoting differentiation partially with respect to x, and let $G(x,y,\lambda)$ denote the Green's function for λ not real. Among the list of 'transforms' $f(x)$, $F(t)$ given in (**1**) were

$$f(x)=G(x,y,\lambda), \qquad F(t)=\phi(y,t)/(\lambda-t);$$
$$f(x)=\partial G(x,y,\lambda)/\partial y, \qquad F(t)=\phi'(y,t)/(\lambda-t).$$

The second result was stated without proof; it can be obtained by a rather tedious limiting process from the corresponding formula for the Sturm–Liouville case. The object of this note is to show that it can be derived very simply, in either the Sturm–Liouville case or the singular case, as a consequence of a general theorem involving functions $F(t)$ for which $tF(t)$ is of integrable square in the appropriate sense. This theorem, together with the transform [8] of § 6 of (**1**), gives a result analogous to the following theorem for Fourier transforms [(**3**), § 3.14, Theorem 68].

If both $f(x)$ and $f'(x)$ belong to L^2, then both $f_1(x)$ and $xf_1(x)$ belong to L^2, and conversely, where f, f_1 are Fourier transforms.

In order to avoid repetition of the preliminaries, I have written this paper as an addendum to (**1**).

2. I consider first the singular case over $(0,\infty)$, in which the unitary transformation from $L^2(0,\infty)$ to $\mathscr{L}^2(-\infty,\infty)$ is given by (1.3) and (1.4) of (**1**), i.e.

$$\int\limits_0^w F(t)\,dk(t)=\int\limits_0^\infty f(x)\,dx\int\limits_0^w \phi(x,t)\,dk(t), \tag{2.1}$$

$$\int\limits_0^x f(y)\,dy=\int\limits_{-\infty}^\infty F(t)\,dk(t)\int\limits_0^x \phi(y,t)\,dy, \tag{2.2}$$

for any $f\in L^2$ and its transform $F\in\mathscr{L}^2$.

THEOREM 1. *Suppose that $F, tF \in \mathscr{L}^2$, that their transforms are $f, h \in L^2$, and write*

$$g(x) = \int_{-\infty}^{\infty} F(t)\phi(x,t)\,dk(t). \tag{2.3}$$

Then $g'(x)$ is absolutely continuous over any interval $(0,a)$ with $a > 0$,

$$g''(x) = q(x)g(x) - h(x),$$

and $g(x) = f(x)$ almost everywhere $(x > 0)$. Also

$$A(g) = g(a)\cos\alpha + g'(a)\sin\alpha = 0. \tag{2.4}$$

Throughout, $\lambda = u+iv$ will be a parameter with $v > 0$, and K a constant, not necessarily the same at each appearance, or a function depending only on the variables indicated.

Since $\phi(x,t)/(\lambda-t)$, $G(x,y,\lambda)$ are transforms, we have, by the Parseval relation and (1) (6.8) with $\mu = \bar{\lambda}$,

$$\int_{-\infty}^{\infty} |F(t)\phi(x,t)|\,dk(t) \leqslant \left\{\int_{-\infty}^{\infty} |(\lambda-t)F(t)|^2\,dk(t) \int_{-\infty}^{\infty} |\phi(x,t)(\lambda-t)^{-1}|^2\,dk(t)\right\}^{\frac{1}{2}}$$
$$\leqslant -K(\lambda)\operatorname{im}\{G(x,x,\lambda)\}. \tag{2.5}$$

Thus $g(x)$ is defined for all $x \geqslant 0$. By (2.2), for any $c \geqslant 0$,

$$\int_c^x (x-t)h(t)\,dt = \int_{-\infty}^{\infty} tF(t)\,dk(t) \int_c^x (x-y)\phi(y,t)\,dy$$
$$= \int_{-\infty}^{\infty} F(t)\,dk(t) \int_c^x (x-y)\{q(y)\phi(y,t) - \phi''(y,t)\}\,dy$$
$$= \int_c^x (x-y)q(y)\,dy \int_{-\infty}^{\infty} F(t)\phi(y,t)\,dk(t) -$$
$$- \int_{-\infty}^{\infty} F(t)\{\phi(x,t) - \phi(c,t) - (x-c)\phi'(c,t)\}\,dk(t).$$

The first inversion is justified as follows if, for instance, $x > c$.

$$\int_c^x d\xi \int_{-\infty}^{\infty} \left|tF \int_c^{\xi} \phi(y,t)\,dy\right| dk \leqslant K \int_c^x \left\{\int_{-\infty}^{\infty} \left|\int_c^{\xi} \phi(y,t)\,dy\right|^2 dk\right\}^{\frac{1}{2}} d\xi$$
$$\leqslant K \int_c^x \left\{\int_c^{\xi} dx\right\}^{\frac{1}{2}} d\xi,$$

when we remember that the transform of a function which is unity for $c \leqslant x \leqslant \xi$ and vanishes otherwise is

$$\int_c^{\xi} \phi(x,t)\,dx.$$

The integral involved in the second inversion is, by (2.5), dominated by

$$-K(\lambda)\int\limits_c^x (x-y)|q(y)|\operatorname{im}\{G(y,y,\lambda)\}\,dy \leqslant K(x,\lambda)$$

since the Green's function is continuous. It follows that

$$g(x) = \int\limits_{-\infty}^{\infty} F(t)\phi(c,t)\,dk(t)+(x-c)\int\limits_{-\infty}^{\infty} F(t)\phi'(c,t)\,dk(t)+$$
$$+\int\limits_c^x (x-y)\{q(y)g(y)-h(y)\}\,dy.$$

All the results stated in the theorem follow from this integral equation.

3. Theorem 2. *For given y and λ not real, the transform of*

$$\frac{\partial G(x,y,\lambda)}{\partial y} \quad \textit{is} \quad \frac{\phi'(y,t)}{\lambda-t}.$$

By (**1**) (6.6), for any $f \in L^2$,

$$\Phi(y,\lambda,f) = \int\limits_0^{\infty} G(x,y,\lambda)f(x)\,dx = \int\limits_{-\infty}^{\infty} \frac{F(t)\phi(y,t)\,dk(t)}{\lambda-t}.$$

Then $F(t) \in \mathscr{L}^2$, and $F(t)/(\lambda-t)$ satisfies the conditions of Theorem 1. Hence

$$\Phi'(y,\lambda,f) = \int\limits_{-\infty}^{\infty} \frac{F(t)\phi'(y,t)\,dk(t)}{\lambda-t}, \tag{3.1}$$

which implies that $F(t)\phi'(y,t)/(\lambda-t) \in \mathscr{L}$ for every $F \in \mathscr{L}^2$. Thus

$$\phi'(y,t)/(\lambda-t) \in \mathscr{L}^2.$$

As in (**1**) § 6, $$\frac{\partial G(x,y,\lambda)}{\partial y} = \begin{cases} \psi(x,\lambda)\phi'(y,\lambda) & (y<x), \\ \phi(x,\lambda)\psi'(y,\lambda) & (y>x), \end{cases}$$

when $\psi(x,\lambda) \in L^2$; hence $\partial G/\partial y \in L^2$ for fixed y. Now

$$\Phi(y,\lambda,f) = \psi(y,\lambda)\int\limits_0^y \phi(x,\lambda)f(x)\,dx + \phi(y,\lambda)\int\limits_y^{\infty} \psi(x,\lambda)f(x)\,dx.$$

Thus, clearly, $$\Phi'(y,\lambda,f) = \int\limits_{-\infty}^{\infty} \frac{\partial G(x,y,\lambda)}{\partial y} f(x)\,dx. \tag{3.2}$$

Let $\Gamma(x,y)$ denote the transform of $\phi'(y,t)/(\lambda-t)$. Then by the Parseval relation (3.1), (3.2), we have

$$\int\limits_0^\infty \left\{\Gamma(x,y)-\frac{\partial G(x,y,\lambda)}{\partial y}\right\} f(x)\,dx = 0$$

for every $f \in L^2$, which proves the result stated.

It is now clear that the relation (1) (6.8) may be differentiated partially with respect to x and ξ. For example

$$\int\limits_0^\infty G_x(x,\xi,\lambda)G_y(y,\xi,\mu)\,d\xi = \int\limits_{-\infty}^\infty \frac{\phi'(x,t)\phi'(y,t)\,dk(t)}{(\lambda-t)(\mu-t)}$$

$$= \frac{G_{xy}(x,y,\lambda)-G_{xy}(x,y,\mu)}{\mu-\lambda}, \tag{3.3}$$

but, if $y = x$, the last expression must be replaced by

$$\frac{1}{\lambda-\mu}\{\psi'(x,\mu)\phi'(x,\mu)-\psi'(x,\lambda)\phi'(x,\lambda)\}. \tag{3.4}$$

4. Similar results hold for the Sturm–Liouville system for the interval (a,b), eigenfunctions $\psi_n(x)$, and boundary conditions

$$\psi_n(a)\cos\alpha+\psi_n'(a)\sin\alpha = \psi_n(b)\cos\beta+\psi_n'(b)\sin\beta = 0.$$

Let (λ_n) be the sequence of eigenvalues, and, for every $f \in L^2(a,b)$, write

$$f_n = \int\limits_a^b f(x)\psi_n(x)\,dx.$$

Corresponding to (2.1), (2.2) we have for any $f \in L^2(a,b)$ $(a \leqslant x \leqslant b)$,

$$\sum_{\lambda_n \leqslant w} f_n = \int\limits_a^b f(x) \sum_{\lambda_n \leqslant w} \psi_n(x)\,dx, \tag{4.1}$$

$$\int\limits_a^x f(t)\,dt = \sum_{n=0}^\infty f_n \int\limits_a^x \psi_n(t)\,dt. \tag{4.2}$$

The analogue of Theorem 1 is as follows.

THEOREM 3. *Suppose that*

$$\sum_{n=0}^\infty |\lambda_n f_n|^2 < \infty, \qquad g(x) = \sum_{n=0}^\infty f_n\psi_n(x),$$

and that f, h are the elements of $L^2(a,b)$ so defined with generalized Fourier coefficients (f_n), $(\lambda_n f_n)$. Then $g'(x)$ is absolutely continuous for $a \leqslant x \leqslant b$, $g''(x) = q(x)g(x)-h(x)$, $g(x) = f(x)$, almost everywhere in (a,b). Also

$$g(a)\cos\alpha+g'(a)\sin\alpha = g(b)\cos\beta+g'(b)\sin\beta = 0.$$

In his development of the theory of the differential equation for the interval (a, b) and in the singular case over $(0, \infty)$ it is assumed by Titchmarsh (**2**) that $q(x)$ is continuous for $a \leqslant x \leqslant b$ or $x \geqslant 0$ respectively. It is quite easy to show that the results of this and the previous paper hold if these conditions are relaxed to $q \in L(a, b)$ and $q \in L(0, c)$ for every $c > 0$.

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ON SOLUTIONS OF THE NON-HOMOGENEOUS FORM OF MILNE'S FIRST INTEGRAL EQUATION

By I. W. BUSBRIDGE (*Oxford*)

[Received 23 March 1955]

1. Introduction

THE integral equation†

$$\left.\begin{aligned} J(\tau) &= \tfrac{1}{2}\chi \int\limits_0^\infty E_1(|t-\tau|)J(t)\,dt + B(\tau) \\ \text{or}\qquad J(\tau) &= \chi\Lambda_\tau\{J(t)\}+B(\tau) \end{aligned}\right\}, \tag{1.1}$$

where Λ is Hopf's operator [see (**5**), or (**7**) chap. II], plays an important part in problems concerning the transfer of radiation through stellar atmospheres with isotropic scattering. It is also of importance, at least with $B(\tau) = 0$, in neutron-diffusion theory.

In (1.1), $J(\tau)$ is the unknown function. However, both in astrophysics and in neutron-diffusion theory, it is the 'emergent intensity' $j(\mu^{-1})$ $(0 < \mu \leqslant 1)$ which is required, where

$$j(s) = \mathfrak{L}_s\{J(t)\} \equiv s \int\limits_0^\infty J(t)e^{-st}\,dt \quad (\operatorname{re} s > 0). \tag{1.2}$$

When $j(s)$ is known, $J(\tau)$ can be found by inverting the Laplace transform, and I shall therefore aim at finding $j(s)$ or $j(\mu^{-1})$.

In (**1**), Ambartsumian has found $j(\mu^{-1})$ for the homogeneous equation by a method depending on the solution of an 'auxiliary equation', viz. (1.1) with $B(\tau) = e^{-\sigma\tau}$ $(\sigma > 0)$.‡ This method is extended in §§ 4–6 to give $j(\mu^{-1})$ when $B(\tau)$ is any polynomial and also when $B(\tau) = E_n(\tau)$. This enables the solution corresponding, for example, to

$$B(\tau) = A_0+A_1\tau+A_2E_2(\tau)+\ldots+A_nE_n(\tau)$$

to be written down, a form which is often to be preferred to a polynomial as an approximation to $B(\tau)$ in practical work. [Cf (**7**), chap. V.] The

† $E_n(\tau)$ is the nth exponential integral defined by

$$E_n(\tau) = \int\limits_1^\infty e^{-x\tau}x^{-n}\,dx \quad (n = 1, 2, \ldots).$$

For its properties, see the appendices of (**2**) and (**7**).

‡ For an account of Ambartsumian's method when $\chi = 1$, see (**7**), § 28.

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result for a polynomial has recently been found by S. S. Huang (**6**) by guessing the form for $j(\mu^{-1})$; that for $E_n(\tau)$ is new. A fundamental role is played by the auxiliary equation and this is investigated in § 3.

In astrophysical problems, the constant χ in (1.1) is usually such that $0 < \chi < 1$ (the so-called 'non-conservative' case) and I shall develop the analysis rigorously for this case. The results hold in modified form when $\chi = 1$, but justification of the processes used becomes lengthy and difficult. Since the main interest of the paper lies in the method, I have simply stated the results for $\chi = 1$ in § 6, indicating the way in which modifications arise. These results can all be verified by substituting $j(s)$ into the equation which is the Laplace transform of (1.1).

Before developing Ambartsumian's method, it is necessary to consider the uniqueness of solutions of (1.1). This is done in § 2.

2. Uniqueness

If $J_1(\tau)$, $J_2(\tau)$ are two solutions of (1.1) and if $J(\tau) = J_1(\tau)-J_2(\tau)$, then

$$J(\tau) = \chi\Lambda_\tau\{J(t)\}. \tag{2.1}$$

When $0 < \chi < 1$, it is well known [see, for example, (**2**) § 88, or (**4**)] that (2.1) has a solution $J(\tau)$ which is $O(e^{k\tau})$ for large τ, where k^{-1} $(0 < k < 1)$ is a zero of

$$T(\mu) = 1-\tfrac{1}{2}\chi\mu\ln\frac{\mu+1}{\mu-1}. \tag{2.2}$$

The corresponding function $j(\mu^{-1})$ is given by

$$j(\mu^{-1}) = \frac{CH(\mu)}{1-k\mu}, \tag{2.3}$$

where $H(\mu)$ is the solution of

$$H(\mu) = 1+\tfrac{1}{2}\chi\mu H(\mu)\int\limits_0^1 \frac{H(x)}{x+\mu}\,dx \tag{2.4}$$

which is regular for $\operatorname{re}\mu > 0$,† and C is an arbitrary constant. The solution $J(\tau)$ is bounded near $\tau = 0$. Thus, if solutions of (1.1) are considered which are $O(e^{k\tau})$ for large τ and $O(1)$ for small τ, and if $j_1(\mu^{-1})$ is one value of $j(\mu^{-1})$, then

$$j_1(\mu^{-1})+CH(\mu)/(1-k\mu)$$

is also a value for any C.

We can, however, prove that, if $J(\tau)$ satisfies (2.1) and is not too large

† $H(\mu)$ will always have this meaning. When $\chi = 1$, there is only one solution of (2.4), but, when $0 < \chi < 1$, there is a second solution with a pole at $\mu = k^{-1}$. See (**3**) or (**2**), Chap. V.

at infinity, then $J(\tau) \equiv 0$. By means of a modified Wiener–Hopf method [cf. (**7**), § 29] I have proved the following theorem, which covers all the cases considered here:

THEOREM I A. *If* $0 < \chi < 1$ *and if* $J(\tau)$ *is a continuous solution of* (2.1) *for* $\tau > 0$ *such that*

$$J(\tau) = O(\ln \tau^{-1}) \quad as\ \tau \to +0, \tag{2.5}$$

$$J(\tau) = o(e^{\theta\tau}) \quad as\ \tau \to \infty \tag{2.6}$$

for every $\theta > 0$, *then* $J(\tau) \equiv 0$.

The proof of this is long and will be omitted, but the following more restricted version is easily proved and it is of interest because it is false when $\chi = 1$.

THEOREM I B. *If* $0 < \chi < 1$ *and if* $J(\tau)$ *is a continuous solution of* (2.1) *for* $\tau > 0$ *such that* $J(\tau)$ *is* $O(1)$ *as* $\tau \to +0$ *and* $J(\tau)$ *is* $O(\tau)$ *as* $\tau \to \infty$, *then* $J(\tau) \equiv 0$.

Under these conditions there is a constant A such that, for all τ,

$$|J(\tau)| \leqslant A(\tau+1). \tag{2.7}$$

Since the kernel of Λ is positive, it follows that

$$-A\Lambda_\tau\{t+1\} \leqslant \Lambda_\tau\{J(t)\} \leqslant A\Lambda_\tau\{t+1\}. \tag{2.8}$$

By (**7**), formulae (14.23) and (14.24),

$$\begin{aligned}\Lambda_\tau\{t+1\} &= \tau+1-\tfrac{1}{2}[E_2(\tau)-E_3(\tau)] \\ &\leqslant \tau+1.\end{aligned} \tag{2.9}$$

Hence (2.8) gives

$$-A(\tau+1) \leqslant \Lambda_\tau\{J(t)\} \leqslant A(\tau+1),$$

i.e., by (2.1), $$|J(\tau)| \leqslant A\chi(\tau+1).$$

Repeating the argument, we have

$$|J(\tau)| \leqslant A\chi^n(\tau+1)$$

for any positive integer n and hence

$$J(\tau) \equiv 0.$$

When $\chi = 1$, the uniqueness has been considered by Hopf in (**5**), Theorem V. He has, in effect, proved

THEOREM II. *Every solution of* (2.1), *with* $\chi = 1$, *which has a finite lower bound is of the form* $CJ^*(\tau)$, *where*

$$j^*(\mu^{-1}) = \mathfrak{L}_{1/\mu}\{J^*(t)\} = H(\mu). \tag{2.10}$$

For large τ, $J^*(\tau)$ *is* $O(\tau)$ *and it is* $O(1)$ *as* $\tau \to +0$.

3. The auxiliary equation

This is the equation

$$J(\tau,\sigma) = \chi\Lambda_\tau\{J(t,\sigma)\}+e^{-\sigma\tau} \quad (\sigma \geqslant 0). \tag{3.1}$$

We shall suppose that $0 < \chi < 1$ and we shall use $J(\tau,\sigma)$ to denote the bounded solution of (3.1). The following theorem proves that this exists and is unique.

THEOREM III. *If $0 < \chi < 1$, the equation (3.1) has a unique solution which is continuous for $\tau > 0$ and is bounded for $\tau \geqslant 0$. This is the N-solution (Neumann series), which converges for $\tau \geqslant 0$.*

The N-solution of (3.1) is

$$J(\tau,\sigma) = \sum_{n=0}^{\infty} \chi^n \Lambda_\tau^n\{e^{-\sigma t}\} \tag{3.2}$$

if this is convergent. Since

$$\Lambda_\tau\{1\} = 1-\tfrac{1}{2}E_2(\tau) \leqslant 1, \tag{3.3}$$

therefore $$0 < \Lambda_\tau^n\{e^{-\sigma t}\} \leqslant \Lambda_\tau^n\{1\} \leqslant 1.$$

Hence the series (3.2) converges uniformly for all $\tau \geqslant 0$ and

$$0 < J(\tau,\sigma) \leqslant (1-\chi)^{-1}.$$

It is easily proved that, if $f(\tau)$ is continuous for $\tau > 0$ and bounded for $\tau \geqslant 0$, then $\Lambda_\tau\{f(t)\}$ is also continuous for $\tau > 0$. Hence each term in (3.2) is continuous for $\tau > 0$ and so $J(\tau,\sigma)$ is continuous. The uniqueness follows from Theorem I.

We now prove some properties of $J(\tau,\sigma)$.

LEMMA I. *If $0 < \chi < 1$ and if $s > 0$, $\sigma > 0$, then*

$$\int_0^\infty e^{-s\tau}J(\tau,\sigma)\,d\tau = \int_0^\infty e^{-\sigma\tau}J(\tau,s)\,d\tau. \tag{3.4}$$

Since the terms of (3.2) are positive and their sum is bounded, we may multiply by $e^{-s\tau}$ and integrate term by term over $(0,\infty)$. Hence

$$\int_0^\infty e^{-s\tau}J(\tau,\sigma)\,d\tau = \sum_{n=0}^{\infty}\chi^n \int_0^\infty e^{-s\tau}\Lambda_\tau^n\{e^{-\sigma t}\}\,d\tau$$

$$= \sum_{n=0}^{\infty}\chi^n \int_0^\infty e^{-\sigma t}\Lambda_t^n\{e^{-s\tau}\}\,dt$$

because the kernel of the operator Λ, and therefore also that of the operator Λ^n, is symmetrical in t and τ. From this (3.4) follows.

LEMMA II. *If* $0 < \chi < 1$,

$$\int\limits_1^\infty J(\tau,\sigma)\frac{d\sigma}{\sigma}$$

exists and is continuous for $\tau > 0$. *It is* $O(1)$ *as* $\tau \to \infty$ *and* $O(\ln \tau^{-1})$ *as* $\tau \to +0$.

Since the terms of (3.2) are positive and since the integrands of all the repeated integrals are positive, we have

$$\int\limits_1^\infty J(\tau,\sigma)\frac{d\sigma}{\sigma} = \sum_{n=0}^{\infty} \chi^n \Lambda_\tau^n \left\{ \int\limits_1^\infty e^{-\sigma t}\frac{d\sigma}{\sigma} \right\} = \sum_{n=0}^{\infty} \chi^n \Lambda_\tau^n \{E_1(t)\} \tag{3.5}$$

if this series is convergent. Let

$$\lambda_1(\tau) = \Lambda_\tau\{E_1(t)\}. \tag{3.6}$$

Then from (**7**), formulae (14.53) and (37.41)–(37.44), it follows that $\lambda_1(\tau)$ is continuous for $\tau > 0$ and $\lambda_1(\tau) \to \ln 2$ as $\tau \to +0$, $\lambda_1(\tau) \to 0$ as $\tau \to \infty$. Hence there is a constant L such that

$$0 \leqslant \lambda_1(\tau) \leqslant L \quad \text{for } \tau \geqslant 0. \tag{3.7}$$

It follows from (3.7) and (3.3) that, if $n \geqslant 1$,

$$0 < \Lambda_\tau^n\{E_1(t)\} = \Lambda_\tau^{n-1}\{\lambda_1(t)\} \leqslant L$$

for $\tau \geqslant 0$. Thus the series (3.5) converges uniformly for $\tau \geqslant 0$ and it is the sum of $E_1(\tau)$ and a function which is continuous for $\tau > 0$ and bounded for $\tau \geqslant 0$. Since $E_1(\tau)$ is $O(\ln \tau^{-1})$ as $\tau \to +0$ and $o(1)$ as $\tau \to \infty$, the stated properties follow.

LEMMA III. *If* $0 < \chi < 1$, $(\partial/\partial\tau)J(\tau,\sigma)$ *exists and is continuous for* $\tau > 0$ *and it is* $O(\ln \tau^{-1})$ *as* $\tau \to +0$ *and* $O(1)$ *as* $\tau \to \infty$.

The proof of this, which is rather longer, will be given only in outline. In (**5**) [see p. 34] Hopf has proved that

$$\frac{d}{d\tau}\Lambda_\tau\{f(t)\} = \Lambda_\tau\{f'(t)\} + \tfrac{1}{2}E_1(\tau)f(0) \tag{3.8}$$

provided that $f'(\tau)$ exists and is continuous for $\tau > 0$ and is such that

$$f'(\tau) = O(\ln \tau^{-1}) \quad \text{as } \tau \to 0, \qquad f'(\tau) = O(e^{\theta\tau}) \quad \text{as } \tau \to \infty,$$

where $\theta < 1$, and provided that $f(0) = \lim\limits_{\tau \to +0} f(\tau)$ exists. Hence

$$\frac{\partial}{\partial\tau}\Lambda_\tau\{e^{-\sigma t}\} = -\sigma\Lambda_\tau\{e^{-\sigma t}\} + \tfrac{1}{2}E_1(\tau) = f_1(\tau) + \tfrac{1}{2}E_1(\tau), \tag{3.9}$$

where $f_1(\tau)$ is continuous for $\tau > 0$ and

$$|f_1(\tau)| \leqslant \sigma. \tag{3.10}$$

Assuming that
$$\frac{\partial}{\partial\tau}\Lambda_\tau^n\{e^{-\sigma t}\} = f_n(\tau)+k_n E_1(\tau), \tag{3.11}$$

where $f_n(\tau)$ is continuous for $\tau > 0$ and†

$$|f_n(\tau)| \leqslant \sigma+\tfrac{1}{2}(n-1)L, \qquad 0 < k_n \leqslant \tfrac{1}{2}, \tag{3.12}$$

and applying (3.8) with $f(\tau) = \Lambda_\tau^n\{e^{-\sigma t}\}$, it is easily proved that (3.11) and (3.12) are true with n replaced by $n+1$. Hence the truth for any n follows from (3.9) and (3.10) by induction.

Let $\tau \geqslant \delta > 0$. Then by (3.11) and (3.12)

$$\left|\chi^n \frac{\partial}{\partial\tau}\Lambda_\tau^n\{e^{-\sigma t}\}\right| \leqslant [\sigma+\tfrac{1}{2}(n-1)L]\chi^n+\tfrac{1}{2}E_1(\delta)\chi^n = (A+Bn)\chi^n,$$

where A and B are constants. Hence the series

$$\sum_{n=0}^{\infty} \chi^n \frac{\partial}{\partial\tau}\Lambda_\tau^n\{e^{-\sigma t}\}$$

converges uniformly for $\tau \geqslant \delta$ and term-by-term differentiation is justified. Also, by (3.11),

$$\frac{\partial}{\partial\tau}J(\tau,\sigma) = -\sigma e^{-\sigma\tau}+\sum_{n=1}^{\infty}\chi^n f_n(\tau)+E_1(\tau)\sum_{n=1}^{\infty}\chi^n k_n,$$

and this is $O(\ln\tau^{-1})$ as $\tau \to +0$ and $O(1)$ as $\tau \to \infty$.

We now use Ambartsumian's method to prove

THEOREM IV. *If* $0 < \chi < 1$, *then, for* $\mu \geqslant 0$,

$$j(\mu^{-1},\sigma) = \mathfrak{L}_{1/\mu}\{J(t,\sigma)\} = \frac{H(\mu)H(\sigma^{-1})}{1+\mu\sigma}. \tag{3.13}$$

By Lemma III, $J(\tau,\sigma)$ satisfies the conditions for Hopf's differentiation formula (3.8). Hence, differentiating (3.1), we have

$$\frac{\partial}{\partial\tau}J(\tau,\sigma) = \chi\Lambda_\tau\left\{\frac{\partial}{\partial t}J(t,\sigma)\right\}+\tfrac{1}{2}\chi J(0,\sigma)E_1(\tau)-\sigma e^{-\sigma\tau}, \tag{3.14}$$

and therefore, by (3.1),

$$\frac{\partial}{\partial\tau}J(\tau,\sigma)+\sigma J(\tau,\sigma) = \chi\Lambda_\tau\left\{\frac{\partial}{\partial t}J(t,\sigma)+\sigma J(t,\sigma)\right\}+\tfrac{1}{2}\chi J(0,\sigma)E_1(\tau). \tag{3.15}$$

By Lemma II, we can multiply (3.1) by σ^{-1} and integrate over $(1,\infty)$.

† L is the constant in (3.7).

Replacing σ by x, we have†

$$\int_1^\infty J(\tau,x)\frac{dx}{x} = \chi\Lambda_\tau\left\{\int_1^\infty J(t,x)\frac{dx}{x}\right\}+E_1(\tau). \tag{3.16}$$

From (3.15) and (3.16) we get

$$\frac{\partial}{\partial\tau}J(\tau,\sigma)+\sigma J(\tau,\sigma)-\tfrac{1}{2}\chi J(0,\sigma)\int_1^\infty J(\tau,x)\frac{dx}{x}$$
$$= \chi\Lambda_\tau\left\{\frac{\partial}{\partial t}J(t,\sigma)+\sigma J(t,\sigma)-\tfrac{1}{2}\chi J(0,\sigma)\int_1^\infty J(t,x)\frac{dx}{x}\right\}.$$

By Theorem III and Lemmas II and III, the function on the left-hand side is continuous for $\tau > 0$ and it is $O(\ln\tau^{-1})$ as $\tau \to +0$ and $O(1)$ as $\tau \to \infty$. Hence, by Theorem I A, it is identically zero. Thus

$$\frac{\partial}{\partial\tau}J(\tau,\sigma)+\sigma J(\tau,\sigma) = \tfrac{1}{2}\chi J(0,\sigma)\int_1^\infty J(\tau,x)\frac{dx}{x}. \tag{3.17}$$

Let $\qquad R(s,\sigma) = s^{-1}\mathfrak{L}_s\{J(t,\sigma)\} = \int_0^\infty e^{-st}J(t,\sigma)\,dt \quad (s>0). \qquad (3.18)$

Then, by Lemma I, $\qquad R(s,\sigma) = R(\sigma,s). \qquad (3.19)$

If we operate on (3.17) by $s^{-1}\mathfrak{L}_s$ $(s > 0)$ and integrate the first term by parts, we get

$$-J(0,\sigma)+sR(s,\sigma)+\sigma R(s,\sigma) = \tfrac{1}{2}\chi J(0,\sigma)\int_1^\infty R(s,x)\frac{dx}{x}$$

and therefore

$$(s+\sigma)R(s,\sigma) = J(0,\sigma)\left\{1+\tfrac{1}{2}\chi\int_1^\infty R(s,x)\frac{dx}{x}\right\}. \tag{3.20}$$

Now put $\tau = 0$ in (3.1). Then

$$\begin{aligned} J(0,\sigma) &= \tfrac{1}{2}\chi\int_0^\infty E_1(t)J(t,\sigma)\,dt+1 \\ &= \tfrac{1}{2}\chi\int_0^\infty J(t,\sigma)\,dt\int_1^\infty e^{-xt}\frac{dx}{x}+1 \\ &= 1+\tfrac{1}{2}\chi\int_1^\infty R(x,\sigma)\frac{dx}{x} \\ &= 1+\tfrac{1}{2}\chi\int_1^\infty R(\sigma,x)\frac{dx}{x}, \end{aligned} \tag{3.21}$$

† This and subsequent inversions of orders of integration are all justified by absolute convergence.

by (3.19). Hence (3.20) becomes

$$(s+\sigma)R(s,\sigma) = J(0,\sigma)J(0,s). \tag{3.22}$$

Substituting for $R(\sigma,x)$ in (3.21), we have

$$J(0,\sigma) = 1+\tfrac{1}{2}\chi J(0,\sigma)\int\limits_{1}^{\infty}\frac{J(0,x)\,dx}{x(x+\sigma)}$$

and, if we put $\sigma = \mu^{-1}$, $x = u^{-1}$, we get

$$J(0,\mu^{-1}) = 1+\tfrac{1}{2}\chi\mu J(0,\mu^{-1})\int\limits_{0}^{1}\frac{J(0,u^{-1})}{u+\mu}\,du. \tag{3.23}$$

Since $J(0,\mu^{-1})$ is bounded for $\mu \geqslant 0$, it must be identical with the solution of (2.4) which is bounded for $\mu \geqslant 0$, i.e. $H(\mu)$. Hence

$$J(0,\mu^{-1}) = H(\mu). \tag{3.24}$$

From (3.18), (3.22), and (3.24), the theorem as stated follows.

4. The exact solution when $B(\tau)$ is a polynomial

We shall now consider the equations

$$J_n(\tau) = \chi\Lambda_\tau\{J_n(t)\}+\frac{a_0\tau^n}{n!}+\frac{a_1\tau^{n-1}}{(n-1)!}+...+a_n \tag{4.1}$$

for $n = 0, 1, 2,...$ and $0 < \chi < 1$. We shall look for functions $J_n(\tau)$ which tend to finite limits $J_n(0)$ as $\tau \to +0$, which satisfy (2.6), and whose derivatives $J'_n(\tau)$ exist, are continuous for $\tau > 0$ and satisfy (2.5) and (2.6). Then the conditions for Hopf's formula (3.8) are satisfied, and we have

$$J'_n(\tau) = \chi\Lambda_\tau\{J'_n(t)\}+\tfrac{1}{2}\chi J_n(0)E_1(\tau)+\frac{a_0\tau^{n-1}}{(n-1)!}+...+a_{n-1}. \tag{4.2}$$

When $n = 0$, the polynomial terms are missing. In (4.1) change n into $n-1$ and subtract from (4.2). Then

$$J'_n(\tau)-J_{n-1}(\tau) = \chi\Lambda_\tau\{J'_n(t)-J_{n-1}(t)\}+\tfrac{1}{2}\chi J_n(0)E_1(\tau), \tag{4.3}$$

and this holds for $n = 0$ if $J_{-1}(\tau)$ is defined to be identically zero. From (4.3) subtract $\frac{1}{2}\chi J_n(0)$ times (3.16); then

$$J'_n(\tau)-J_{n-1}(\tau)-\tfrac{1}{2}\chi J_n(0)\int\limits_{1}^{\infty}J(\tau,x)\frac{dx}{x}$$
$$= \chi\Lambda_\tau\left\{J'_n(t)-J_{n-1}(t)-\tfrac{1}{2}\chi J_n(0)\int\limits_{1}^{\infty}J(t,x)\frac{dx}{x}\right\}.$$

The function on the left-hand side satisfies the conditions of Theorem I A and it is therefore identically zero. Hence

$$J_n'(\tau)-J_{n-1}(\tau) = \tfrac{1}{2}\chi J_n(0)\int\limits_1^\infty J(\tau,x)\,\frac{dx}{x}. \tag{4.4}$$

Let
$$j_n(s) = \mathfrak{L}_s\{J_n(t)\} \quad (s > 0). \tag{4.5}$$

If we operate on (4.4) by $s^{-1}\mathfrak{L}_s$, integrate the first term on the left by parts, and invert the order of integration on the right, we get

$$-J_n(0)+j_n(s)-s^{-1}j_{n-1}(s) = \tfrac{1}{2}\chi J_n(0)\int\limits_1^\infty R(s,x)\,\frac{dx}{x},$$

and hence, by (3.21),

$$j_n(s)-s^{-1}j_{n-1}(s) = J_n(0)J(0,s). \tag{4.6}$$

When $n = 0$, this is
$$j_0(s) = J_0(0)J(0,s). \tag{4.7}$$

From (4.6) and (4.7) we have

$$j_n(s) = J(0,s)\{J_n(0)+s^{-1}J_{n-1}(0)+...+s^{-n}J_0(0)\}, \tag{4.8}$$

i.e., by (3.24),

$$j_n(\mu^{-1}) = H(\mu)\{J_n(0)+\mu J_{n-1}(0)+...+\mu^n J_0(0)\}. \tag{4.9}$$

By means of (4.8) we can verify the conditions assumed above for $J_n(\tau)$. It is easily verified that the proofs of Theorem III and Lemma III hold for $\sigma = 0$. Hence $J(\tau, 0)$ is bounded for $\tau \geqslant 0$ and $(d/d\tau)J(\tau,0)$ is continuous for $\tau > 0$, is $O(\ln \tau^{-1})$ as $\tau \to +0$ and $O(1)$ as $\tau \to \infty$. Comparing (3.1) and (4.1) (with $n = 0$), we see that

$$J_0(\tau) = a_0\, J(\tau, 0), \tag{4.10}$$

and hence by (4.7) the Laplace transform of $J(\tau, 0)$ is a constant multiple of $J(0, s)$. The functions whose Laplace transforms are $s^{-1}J(0,s)$, $s^{-2}J(0,s)$, etc., are therefore constant multiples of

$$\int\limits_0^\tau J(t,0)\,dt, \qquad \int\limits_0^\tau dt\int\limits_0^t J(u,0)\,du, \ \text{ etc.,} \tag{4.11}$$

and these functions all satisfy the assumed conditions. Actually $J_n(\tau)$ is $O(\tau^n)$ for large τ.

It remains to find the constants $J_0(0), J_1(0),\ldots$. From (4.1), with $\tau = 0$

$$J_n(0) = \tfrac{1}{2}\chi \int\limits_0^\infty J_n(t)E_1(t)\,dt + a_n$$

$$= \tfrac{1}{2}\chi \int\limits_0^\infty J_n(t)\,dt \int\limits_1^\infty e^{-xt}\frac{dx}{x} + a_n$$

$$= \tfrac{1}{2}\chi \int\limits_1^\infty j_n(x)\frac{dx}{x^2} + a_n$$

$$= \tfrac{1}{2}\chi \int\limits_0^1 j_n(\mu^{-1})\,d\mu + a_n. \tag{4.12}$$

Let

$$\alpha_n = \int\limits_0^1 \mu^n H(\mu)\,d\mu. \tag{4.13}$$

On substituting from (4.9) into (4.12) we have

$$J_n(0) = \tfrac{1}{2}\chi \sum_{\nu=0}^{n} \alpha_\nu J_{n-\nu}(0) + a_n. \tag{4.14}$$

But, by (2), Chap. V, Theorem 1,

$$\tfrac{1}{2}\chi\alpha_0 = 1-(1-\chi)^{\frac{1}{2}}, \tag{4.15}$$

and so

$$(1-\chi)^{\frac{1}{2}} J_n(0) - \tfrac{1}{2}\chi \sum_{\nu=1}^{n} \alpha_\nu J_{n-\nu}(0) = a_n \quad (n = 0, 1,\ldots). \tag{4.16}$$

Solving, we have

$$J_\nu(0) = (1-\chi)^{-\frac{1}{2}(\nu+1)} \begin{vmatrix} a_\nu & -\frac{1}{2}\chi\alpha_1 & -\frac{1}{2}\chi\alpha_2 & \cdot & \cdot & \cdot & -\frac{1}{2}\chi\alpha_\nu \\ a_{\nu-1} & (1-\chi)^{\frac{1}{2}} & -\frac{1}{2}\chi\alpha_1 & \cdot & \cdot & \cdot & -\frac{1}{2}\chi\alpha_{\nu-1} \\ a_{\nu-2} & 0 & (1-\chi)^{\frac{1}{2}} & \cdot & \cdot & \cdot & -\frac{1}{2}\chi\alpha_{\nu-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_0 & 0 & 0 & \cdot & \cdot & \cdot & (1-\chi)^{\frac{1}{2}} \end{vmatrix}. \tag{4.17}$$

Thus we have proved

THEOREM V. *If $0 < \chi < 1$, the equation (4.1) has a solution $J_n(\tau)$ which is continuous for $\tau > 0$, tends to a finite limit as $\tau \to +0$ and is $O(\tau^n)$ as $\tau \to \infty$. If $j_n(s) = \mathfrak{L}_s\{J_n(t)\}$, then $j_n(\mu^{-1})$ $(\mu \geqslant 0)$ is given by (4.9), where the constants $J_\nu(0)$ are given by (4.17).*

5. The exact solution when $B(\tau) = E_n(\tau)$

We shall denote the solution of (1.1) with $B(\tau) = E_n(\tau)$ by $\Phi_n(\tau)$. When $n = 1$, we have

$$\Phi_1(\tau) = \chi\Lambda_\tau\{\Phi_1(t)\} + E_1(\tau). \tag{5.1}$$

Comparing with (3.15), we see that

$$\Phi_1(\tau) = \frac{2}{\chi J(0,\sigma)}\left\{\frac{\partial}{\partial\tau}J(\tau,\sigma)+\sigma J(\tau,\sigma)\right\}. \tag{5.2}$$

By Lemma III and Theorem III, this is $O(\ln\tau^{-1})$ as $\tau\to+0$ and $O(1)$ as $\tau\to\infty$. Operating on this by $\mathfrak{L}_s$ $(s>0)$ and integrating by parts on the right, we have

$$\phi_1(s) = \mathfrak{L}_s\{\Phi_1(t)\} = \frac{2s}{\chi J(0,\sigma)}\{-J(0,\sigma)+sR(s,\sigma)+\sigma R(s,\sigma)\},$$

where $R(s,\sigma)$ is given by (3.18). By (3.22) this is

$$\phi_1(s) = \frac{2s}{\chi}\{J(0,s)-1\}, \tag{5.3}$$

and so
$$\phi_1(\mu^{-1}) = \frac{2}{\chi\mu}\{H(\mu)-1\}. \tag{5.4}$$

Now consider the equation

$$\Phi_n(\tau) = \chi\Lambda_\tau\{\Phi_n(t)\}+E_n(\tau) \quad (n\geqslant 2), \tag{5.5}$$

and suppose that $\Phi_n(\tau)$ tends to a finite limit $\Phi_n(0)$ as $\tau\to 0$ and satisfies (2.6). Suppose also that $\Phi_n'(\tau)$ exists and is continuous for $\tau>0$ and satisfies (2.5) and (2.6). Then we can differentiate (5.5), giving

$$\Phi_n'(\tau) = \chi\Lambda_\tau\{\Phi_n'(t)\}+\tfrac{1}{2}\chi\Phi_n(0)E_1(\tau)-E_{n-1}(\tau). \tag{5.6}$$

Adding (5.5) with n replaced by $n-1$ and subtracting $\frac{1}{2}\chi\Phi_n(0)$ times (5.1), we have

$$\Phi_n'(\tau)+\Phi_{n-1}(\tau)-\tfrac{1}{2}\chi\Phi_n(0)\Phi_1(\tau)$$
$$= \chi\Lambda_\tau\{\Phi_n'(t)+\Phi_{n-1}(t)-\tfrac{1}{2}\chi\Phi_n(0)\Phi_1(t)\}.$$

The function on the left satisfies the conditions of Theorem I A and so is identically zero. Hence

$$\Phi_n'(\tau)+\Phi_{n-1}(\tau) = \tfrac{1}{2}\chi\Phi_n(0)\Phi_1(\tau). \tag{5.7}$$

Let
$$\phi_n(s) = \mathfrak{L}_s\{\Phi_n(t)\} \quad (s>0). \tag{5.8}$$

Operating on (5.7) by $s^{-1}\mathfrak{L}_s$ and integrating by parts, we have

$$-\Phi_n(0)+\phi_n(s)+s^{-1}\phi_{n-1}(s) = \tfrac{1}{2}\chi s^{-1}\Phi_n(0)\phi_1(s),$$

and hence, by (5.3),

$$\phi_n(s)+s^{-1}\phi_{n-1}(s) = \Phi_n(0)J(0,s) \quad (n\geqslant 2). \tag{5.9}$$

By (5.9) and (5.3),

$$\phi_n(s)$$
$$= J(0,s)\left\{\Phi_n(0)-s^{-1}\Phi_{n-1}(0)+\ldots+(-1)^{n-2}s^{2-n}\Phi_2(0)+(-1)^{n-1}\frac{2}{\chi}s^{2-n}\right\}+$$
$$+(-1)^n\frac{2}{\chi}s^{2-n} \quad (n\geqslant 2). \tag{5.10}$$

Comparing this with (4.8) and the argument which follows, it is seen that $\Phi_n(\tau)$ $(n \geqslant 2)$ satisfies the conditions assumed above. Actually $\Phi_n(\tau)$ is $O(\tau^{n-2})$ for large τ. From (5.10) we have

$$\phi_n(\mu^{-1})$$
$$= H(\mu)\Big\{\Phi_n(0)-\mu\Phi_{n-1}(0)+\ldots+(-1)^{n-2}\mu^{n-2}\Phi_2(0)+(-1)^{n-1}\frac{2}{\chi}\mu^{n-2}\Big\}+$$
$$+(-1)^n\frac{2}{\chi}\mu^{n-2} \quad (n \geqslant 2). \tag{5.11}$$

In order to find the constants $\Phi_\nu(0)$, we put $\tau = 0$ in (5.5) and proceed as in (4.12). Since $E_n(0) = 1/(n-1)$, we get

$$\Phi_n(0) = \tfrac{1}{2}\chi \int\limits_0^1 \phi_n(\mu^{-1})\,d\mu+\frac{1}{n-1}. \tag{5.12}$$

Substituting from (5.11) and using (4.15), this gives us

$$(1-\chi)^{\frac{1}{2}}\Phi_n(0)+\tfrac{1}{2}\chi\alpha_1\,\Phi_{n-1}(0)-\ldots+(-1)^{n-1}\,\tfrac{1}{2}\chi\alpha_{n-2}\,\Phi_2(0)$$
$$= (-1)^{n-1}\alpha_{n-2}+\frac{1+(-1)^n}{n-1} \quad (n \geqslant 2). \tag{5.13}$$

Solving, we have

$$\Phi_\nu(0) = (1-\chi)^{-\frac{1}{2}(\nu-1)}\begin{vmatrix} c_\nu & \frac{1}{2}\chi\alpha_1 & -\frac{1}{2}\chi\alpha_2 & . & . & . & (-1)^{\nu-1}\,\frac{1}{2}\chi\alpha_{\nu-2} \\ c_{\nu-1} & (1-\chi)^{\frac{1}{2}} & \frac{1}{2}\chi\alpha_1 & . & . & . & (-1)^{\nu-2}\,\frac{1}{2}\chi\alpha_{\nu-3} \\ c_{\nu-2} & 0 & (1-\chi)^{\frac{1}{2}} & . & . & . & (-1)^{\nu-3}\,\frac{1}{2}\chi\alpha_{\nu-4} \\ . & . & . & . & . & . & . \\ c_2 & 0 & 0 & . & . & . & (1-\chi)^{\frac{1}{2}} \end{vmatrix}, \tag{5.14}$$

where
$$c_\nu = \frac{1+(-1)^\nu}{\nu-1}+(-1)^{\nu-1}\alpha_{\nu-2} \tag{5.15}$$

and $\nu = 2, 3,\ldots$. Thus we have

THEOREM VI. *If* $0 < \chi < 1$ *and* $n \geqslant 2$, *the equation* (5.5) *has a solution* $\Phi_n(\tau)$ *which is continuous for* $\tau > 0$, *tends to a finite limit as* $\tau \to +0$ *and is* $O(\tau^{n-2})$ *as* $\tau \to \infty$. *If* $\phi_n(s) = \mathfrak{L}_s\{\Phi_n(t)\}$, *then* $\phi(\mu^{-1})$ $(\mu \geqslant 0)$ *is given by* (5.11), *where the constants* $\Phi_\nu(0)$ *are given by* (5.14) *and* (5.15). *When* $n = 1$ *the solution is* $O(\ln\tau^{-1})$ *as* $\tau \to +0$ *and* $O(1)$ *as* $\tau \to \infty$, *and* $\phi_1(\mu^{-1})$ $(\mu > 0)$ *is given by* (5.4).

6. The conservative case $(\chi = 1)$

When $\chi = 1$, we are on the circle of convergence of the N-solution (3.2) of the auxiliary equation. It follows, however, from a theorem of Hopf's [see (5), Theorem VII] that the series converges and so $J(\tau, \sigma)$

still exists. Theorem IV remains true, but it is necessary to use indirect methods to establish the properties of $J(\tau,\sigma)$ which are needed in the course of the proof.

In §§ 4 and 5, an appeal has to be made to Theorem II instead of Theorem I A, and so, for example, (4.4) has to be replaced by

$$J'_n(\tau)-J_{n-1}(\tau) = \tfrac{1}{2}J_n(0)\int\limits_1^\infty J(\tau,x)\frac{dx}{x}+C_n J^*(\tau), \tag{6.1}$$

where $j^*(\mu^{-1}) = H(\mu)$ and C_n is a constant. In place of (4.9) we get

$$\begin{aligned} j_n(\mu^{-1}) & \\ &= H(\mu)\{J_n(0)+\mu J_{n-1}(0)+...+\mu^n J_n(0)+\mu C_n+\mu^2 C_{n-1}+...+\mu^{n+1}C_0\} \\ &= H(\mu)\{J_n(0)+C'_n\mu+C'_{n-1}\mu^2+...+C'_0\mu^{n+1}\}, \end{aligned} \tag{6.2}$$

where $\qquad C'_0 = C_0,\ C'_\nu = C_\nu+J_{\nu-1}(0) \quad (\nu = 1, 2,..., n).$

Equation (4.12) holds with $\chi = 1$, and on substituting from (6.2) we have

$$J_n(0) = \tfrac{1}{2}\{J_n(0)\alpha_0+C'_n\alpha_1+...+C'_0\alpha_{n+1}\}+a_n.$$

But $\alpha_0 = 2$ when $\chi = 1$, and so

$$C'_n\alpha_1+C'_{n-1}\alpha_2+...+C'_0\alpha_{n+1} = -2a_n \quad (n \geqslant 0). \tag{6.3}$$

Hence

$$C'_\nu = -\frac{2}{\alpha_1^{\nu+1}}\begin{vmatrix} a_\nu & \alpha_2 & \alpha_3 & . & . & . & \alpha_{\nu+1} \\ a_{\nu-1} & \alpha_1 & \alpha_2 & . & . & . & \alpha_\nu \\ a_{\nu-2} & 0 & \alpha_1 & . & . & . & \alpha_{\nu-1} \\ . & . & . & . & . & . & . \\ a_0 & 0 & 0 & . & . & . & \alpha_1 \end{vmatrix}. \tag{6.4}$$

Equations (6.2) and (6.4) together give the required solution. The constant $J_n(0)$ is arbitrary.

Similarly it is found that (5.4) has to be replaced by

$$\phi_1(\mu^{-1}) = 2\mu^{-1}\{H(\mu)-1\}+CH(\mu), \tag{6.5}$$

where C is arbitrary. The equation (5.11) is replaced by

$$\begin{aligned} \phi_n(\mu^{-1}) = H(\mu)\{\Phi_n(0)+A_n\mu-A_{n-1}\mu^2+...+(-1)^{n-2}A_2\mu^{n-1}\}+ & \\ +2(-1)^{n-1}\mu^{n-2}[H(\mu)-1] \quad (n \geqslant 2), & \end{aligned} \tag{6.6}$$

where the constants A_ν are given by the equations

$$A_n\alpha_1-A_{n-1}\alpha_2+...+(-1)^{n-2}A_2\alpha_{n-1} = -2c_n \quad (n = 2,3,...) \tag{6.7}$$

and c_ν is given by (5.15). Hence

$$A_\nu = -\frac{2}{\alpha_1^{\nu-1}} \begin{vmatrix} c_\nu & -\alpha_2 & \alpha_3 & . & . & . & (-1)^\nu \alpha_{\nu-1} \\ c_{\nu-1} & \alpha_1 & -\alpha_2 & . & . & . & (-1)^{\nu-1}\alpha_{\nu-2} \\ c_{\nu-2} & 0 & \alpha_1 & . & . & . & (-1)^{\nu-2}\alpha_{\nu-3} \\ . & . & . & . & . & . & . \\ c_2 & 0 & 0 & . & . & . & \alpha_1 \end{vmatrix}. \tag{6.8}$$

The constant $\Phi_n(0)$ in (6.6) is arbitrary.

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POISSON'S PARTIAL DIFFERENCE EQUATION

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1. Introduction

THE equation $\nabla^2 V = -4\pi\rho$ is well known in electrostatics and in problems concerning the steady flow of electric currents in conducting media, and the solutions are given most easily in terms of the appropriate Green's functions. It is convenient to deal with a unit-source function, i.e. we take $\rho = \delta(\mathbf{r})$ where $\delta(x)$ denotes the Dirac delta function and $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$ in three dimensions. Then, for example, a solution of Poisson's equation in three dimensions

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = -4\pi\delta(\mathbf{r}) \tag{1}$$

is $V = r^{-1}$, and a similar solution in two dimensions is $V = -2\log r$.

Instead of considering the problem of current flow in uniform conducting media due to a certain source distribution of current, it is possible to consider the analogous problem of current flow in a two-dimensional or three-dimensional grid of conducting wires. Instead of a partial differential equation, which is nothing more than an equation expressing the conservation of electric current, we get a partial difference equation. This note is concerned with the particular solutions of this type of difference equation which correspond to the solutions mentioned above. In short, the problem is: *given an infinite, uniform, regular grid of wires with current entering at a grid point, what is the distribution of potential and current in the grid?* Or, *What are the appropriate Green's functions for this type of partial difference equation?*

2. The equation and its solutions

We consider first a two-dimensional, infinite Cartesian grid, each element of which has resistance R, with current I entering the grid at the grid point $(0,0)$. Then, if the potential of the grid point (l,m) be $V_{l,m}$ we have immediately the conservation equation

$$4V_{l,m}-V_{l-1,m}-V_{l+1,m}-V_{l,m-1}-V_{l,m+1} = RI\,\delta_{l0}\,\delta_{m0}, \tag{2}$$

where δ_{rs} is the usual Kronecker delta.

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The method we adopt for getting a particular solution is the same as that used in the corresponding partial differential equation; we use a convenient representation for the delta symbol, writing

$$\delta_{l0}\,\delta_{m0} = \frac{1}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} e^{ilx+imy}\,dxdy \tag{3}$$

and now we seek a solution of the form

$$V_{l,m} = \int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} f(x,y)e^{ilx+imy}\,dxdy. \tag{4}$$

Then equation (2) becomes

$$\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} f(x,y)e^{ilx+imy}(4-e^{ix}-e^{-ix}-e^{iy}-e^{-iy})\,dxdy$$

$$= \frac{RI}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} e^{ilx+imy}\,dxdy, \tag{5}$$

and so the solution we want is

$$V_{l,m} = \frac{RI}{4\pi^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \frac{e^{ilx+imy}}{4-2\cos x-2\cos y}\,dxdy \tag{6}$$

$$= \frac{RI}{2\pi^2}\int\limits_{0}^{\pi}\int\limits_{0}^{\pi} \frac{\cos lx\cos my}{2-\cos x-\cos y}\,dxdy$$

$$= \frac{RI}{\pi^2}\int\limits_{0}^{\frac{1}{2}\pi}\int\limits_{0}^{\frac{1}{2}\pi} \frac{\cos 2lx\cos 2my}{\sin^2 x+\sin^2 y}\,dxdy. \tag{7}$$

Such a solution has the correct symmetry in (l, m) but it is a divergent double integral because of the behaviour of the integrand near the origin. The integral behaves like

$$\int\limits_{0}\int\limits_{0} \frac{dxdy}{x^2+y^2} = \int\int \frac{r\,drd\theta}{r^2} = \tfrac{1}{2}\pi\int\limits_{0} \frac{dr}{r}. \tag{8}$$

This logarithmic singularity is not unexpected because of the similar behaviour of the solution in the two-dimensional case. But we are

interested only in differences of potential and, if we normalize the potential at the grid point $(0, 0)$ to be zero, then the solution is

$$V_{l,m} = -\frac{RI}{\pi^2} \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{1-\cos 2lx \cos 2my}{\sin^2 x + \sin^2 y} \, dx dy \tag{9}$$

and the awkward behaviour of the integrand at the origin disappears.

The solution $V_{l,m}$ can be evaluated without recourse to numerical methods by the following tedious but straightforward process. The numerator of the integrand can be expanded into terms of the type $\sin^{2r}x \sin^{2s}y$. Then

$$\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\sin^{2r}x \sin^{2s}y}{\sin^2 x + \sin^2 y} \, dx dy$$

$$= \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{(\sin^{2r}x \pm \sin^{2r}y)\sin^{2s}y \mp \sin^{2(r+s)}y}{\sin^2 x + \sin^2 y} \, dx dy. \tag{10}$$

Now

$$\int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}y}{\sin^2 x + \sin^2 y} \, dx dy = \tfrac{1}{2}\pi \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n-1}y}{\sqrt{(1+\sin^2 y)}} \, dy$$

$$= \tfrac{1}{2}\pi \int_0^{\frac{1}{2}\pi} \frac{\sin^{2n-1}y}{\sqrt{(2-\cos^2 y)}} \, dy = \tfrac{1}{2}\pi \int_0^{\frac{1}{4}\pi} (1-2\sin^2\theta)^{n-1} \, d\theta$$

$$= \tfrac{1}{4}\pi \int_0^{\frac{1}{2}\pi} \cos^{n-1}\phi \, d\phi. \tag{11}$$

But $(\sin^{2r}x \pm \sin^{2s}y)/(\sin^2 x + \sin^2 y)$ can be expanded in powers of $\sin^2 x$ and $\sin^2 y$, the upper sign being chosen for odd r and the lower sign for even r. The integrations are all elementary.

3. The equation in three dimensions

The partial difference equation for a three-dimensional grid is

$$6V_{l,m,n} - V_{l-1,m,n} - V_{l+1,m,n} - V_{l,m-1,n} - V_{l,m+1,n} - V_{l,m,n-1} - V_{l,m,n+1}$$
$$= RI\,\delta_{l0}\,\delta_{m0}\,\delta_{n0} \tag{12}$$

and using

$$\delta_{l0}\,\delta_{m0}\,\delta_{n0} = \frac{1}{8\pi^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{ilx+imy+inz} \, dx dy dz \tag{13}$$

we get, for the cubic grid with a source at the grid point $(0,0,0)$, a solution,

$$V_{l,m,n} = \frac{RI}{8\pi^3}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} \frac{e^{ilx+imy+inz}}{6-2\cos x-2\cos y-2\cos z}\,dxdydz$$

$$= \frac{2RI}{\pi^3}\int\limits_{0}^{\frac{1}{2}\pi}\int\limits_{0}^{\frac{1}{2}\pi}\int\limits_{0}^{\frac{1}{2}\pi} \frac{\cos 2lx\cos 2my\cos 2nz}{\sin^2 x+\sin^2 y+\sin^2 z}\,dxdydz. \tag{14}$$

The general solution involves additive solutions of the homogeneous partial difference equation, but these extra terms do not have the right kind of symmetry behaviour for our particular problem.

The triple integral (14) is convergent since, near the origin, where the integrand diverges, the integral behaves like

$$\int\limits_0\int\limits_0\int\limits_0 \frac{dxdydz}{x^2+y^2+z^2} = 4\pi\int\limits_0 dr. \tag{15}$$

In this case I shall show later that $V_{l,m,n} \to 0$ as l, m, $n \to \infty$ and so it is convenient to consider the grid points at infinity to be earthed, i.e. at zero potential. The situation is similar in the continuous case in three dimensions where it is usual to make the potential at infinity zero, while in two dimensions this is not convenient.

To show that $V_{l,m,n} \to 0$ as $l \to \infty$, we make use of the following lemma [(2) 172].

Let $\int\limits_a^b \psi(\theta)\,d\theta$ exist and, if it is an improper integral, let it be absolutely convergent. Then, as $\lambda \to \infty$,

$$\lim_{\lambda\to\infty}\int\limits_a^b \psi(\theta)\cos\lambda\theta\,d\theta = 0. \tag{16}$$

Now we can write

$$I = \int\limits_0^{\frac{1}{2}\pi}\left\{\int\limits_0^{\frac{1}{2}\pi}\int\limits_0^{\frac{1}{2}\pi} \frac{\cos 2my\cos 2nz}{\sin^2 x+\sin^2 y+\sin^2 z}\,dydz\right\}\cos 2lx\,dx$$

$$= \int\limits_0^{\frac{1}{2}\pi} \psi(x)\cos 2lx\,dx. \tag{17}$$

We want to show that $\int\limits_0^{\frac{1}{2}\pi} \psi(x)\,dx$ exists. This integral is improper,

$\psi(x)$ having a singularity at $x = 0$. But it is absolutely convergent for

$$\int\limits_0^\delta |\psi(x)|\,dx = \int\limits_0^\delta \left| \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{\cos 2my \cos 2nz}{\sin^2 x + \sin^2 y + \sin^2 z}\,dydz \right| dx$$

$$\leqslant \int\limits_0^\delta \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{dxdydz}{\sin^2 x + \sin^2 y + \sin^2 z} < \epsilon \quad \text{for some } \delta \qquad (18)$$

because

$$\int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{dxdydz}{\sin^2 x + \sin^2 y + \sin^2 z}$$

exists. Hence $\lim\limits_{l\to\infty} I = 0$ and so $V_{l,m,n} \to 0$ as l, m, $n \to \infty$.

The potential difference between the origin and the grid points at infinity is

$$V_{0,0,0} = \frac{2RI}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{dxdydz}{\sin^2 x + \sin^2 y + \sin^2 z}, \qquad (19)$$

and we can speak of the resistance between a grid point and infinity in the same way as we speak of the resistance between a spherical electrode embedded in a conducting medium and an electrode at infinity. This resistance is

$$R_g = \frac{V_{0,0,0}}{I} = \frac{2R}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{dxdydz}{\sin^2 x + \sin^2 y + \sin^2 z}. \qquad (20)$$

Now

$$\int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{dxdydz}{\sin^2 x + \sin^2 y + \sin^2 z}$$

$$= \tfrac{1}{2}\pi \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{dxdy}{\sqrt{(\sin^2 x + \sin^2 y)}\sqrt{(1 + \sin^2 x + \sin^2 y)}}$$

$$= \tfrac{1}{2}\pi \int\limits_0^{\frac{1}{2}\pi} \frac{dx}{1+\sin^2 x} \int\limits_0^1 \frac{du}{\sqrt{(1-u^2)}\sqrt{\{1-u^2(1+\sin^2 x)^{-2}\}}},$$

where $u = \dfrac{\sin y\sqrt{(1+\sin^2 x)}}{\sqrt{(\sin^2 x + \sin^2 y)}}$,

$$= \tfrac{1}{2}\pi \int\limits_0^{\frac{1}{2}\pi} \frac{dx}{1+\sin^2 x} \int\limits_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{\{1-\sin^2\phi(1+\sin^2 x)^{-2}\}}}$$

$$= \tfrac{1}{2}\pi \int\limits_0^{\frac{1}{2}\pi} \frac{1}{1+\sin^2 x} K\left(\frac{1}{1+\sin^2 x}\right) dx, \qquad (21)$$

where $K(k)$ is the elliptic integral [(1) 73] defined by

$$K(k) = \int\limits_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}, \tag{22}$$

and so

$$R_g = \frac{R}{\pi^2} \int\limits_0^{\frac{1}{2}\pi} \frac{1}{1+\sin^2 x} K\left(\frac{1}{1+\sin^2 x}\right) dx = \frac{R}{2\pi^2} \int\limits_{\frac{1}{2}}^{1} \frac{K(k)}{\sqrt{\{(1-k)(2k-1)\}}} dk, \tag{23}$$

which can easily be evaluated numerically.

The general expression for $V_{l,m,n}$ will almost certainly involve elliptic integrals.

4. The resistance between two grid points

If there is a source of strength I at the grid point $(0, 0, 0)$ and the grid is earthed at infinity, then the potential of the grid point (l, m, n) is

$$V_{l,m,n} = \frac{2RI}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{\cos 2lx \cos 2my \cos 2nz}{\sin^2 x + \sin^2 y + \sin^2 z} \, dxdydz.$$

If we now place a sink of strength I at the grid point (l, m, n) with the grid earthed at infinity, the potential of the grid point $(0, 0, 0)$ is

$$V'_{0,0,0} = -\frac{2RI}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{\cos 2lx \cos 2my \cos 2nz}{\sin^2 x + \sin^2 y + \sin^2 z} \, dxdydz. \tag{24}$$

Superimposing the two solutions, we get a potential difference between the two grid points

$$\begin{aligned} V_{0,0,0} + V'_{0,0,0} - V_{l,m,n} - V'_{l,m,n} &= 2V_{0,0,0} - 2V_{l,m,n} \\ &= \frac{4RI}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{1-\cos 2lx \cos 2my \cos 2nz}{\sin^2 x + \sin^2 y + \sin^2 z} \, dxdydz, \end{aligned} \tag{25}$$

and so the resistance between the two points is

$$R_{l,m,n} = \frac{4R}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{1-\cos 2lx \cos 2my \cos 2nz}{\sin^2 x + \sin^2 y + \sin^2 z} \, dxdydz \tag{26}$$

with a similar result for the two-dimensional grid.

It is easy to evaluate the resistance between two neighbours on the grid for

$$R_{1,0,0} = \frac{4R}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{1-\cos 2x}{\sin^2 x + \sin^2 y + \sin^2 z}\, dxdydz$$

$$= \frac{8R}{\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{\sin^2 x}{\sin^2 x + \sin^2 y + \sin^2 z}\, dxdydz$$

$$= \frac{8R}{3\pi^3} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \int\limits_0^{\frac{1}{2}\pi} \frac{\sin^2 x + \sin^2 y + \sin^2 z}{\sin^2 x + \sin^2 y + \sin^2 z}\, dxdydz$$

$$= \tfrac{1}{3}R. \tag{27}$$

However, one can obtain this result without knowing the form of $V_{l,m,n}$ by using physical arguments which correspond to the use of symmetry in the evaluation of the above integral. In two dimensions $R_{10} = \frac{1}{2}R$.

5. Asymptotic behaviour of the solutions

We have seen that in two dimensions the potential distribution due to a source at $(0, 0)$ is

$$V_{l,m} = -\frac{RI}{16\pi^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \frac{1-e^{ilx+imy}}{\sin^2 \frac{1}{2}x + \sin^2 \frac{1}{2}y}\, dxdy.$$

We are now interested in the behaviour of $V_{l,m}$ as l, $m \to \infty$. Now

$$V_{l,m} = -\frac{RI}{4\pi^2}\left\{ \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \frac{1-e^{ilx+imy}}{x^2+y^2}\, dxdy + \right.$$

$$\left. + \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} (1-e^{ilx+imy})\left[\frac{1}{4\sin^2 \frac{1}{2}x + 4\sin^2 \frac{1}{2}y} - \frac{1}{x^2+y^2}\right] dxdy \right\}. \tag{28}$$

The integrand in the second term is everywhere finite and so the second integral is of order unity. Thus

$$V_{l,m} = -\frac{RI}{4\pi^2}\left\{ \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \frac{1-e^{ilx+imy}}{x^2+y^2}\, dxdy + O(1)\right\}$$

$$= -\frac{RI}{4\pi^2}\left\{ \int\limits_0^{\pi} \int\limits_0^{2\pi} (1-e^{ir\sqrt{(l^2+m^2)}\cos\theta})r^{-1}\, drd\theta + O(1)\right\}, \tag{29}$$

where we have changed to polar coordinates, the domain of integration omitted making a contribution of $O(1)$. We get therefore

$$V_{l,m} = -\frac{RI}{2\pi}\Big[\int_0^{\pi}\{1-J_0(r\sqrt{(l^2+m^2)})\}r^{-1}\,dr + O(1)\Big]$$

$$= -\frac{RI}{2\pi}\Big[\int_0^{\pi\sqrt{(l^2+m^2)}}\{1-J_0(z)\}z^{-1}\,dz + O(1)\Big]$$

$$= -\frac{RI}{2\pi}\Big[\int_0^{1}\{1-J_0(z)\}z^{-1}\,dz + \int_1^{\pi\sqrt{(l^2+m^2)}}\{1-J_0(z)\}z^{-1}\,dz + O(1)\Big]. \tag{30}$$

The first integral is $O(1)$ and so

$$V_{l,m} = -\frac{RI}{2\pi}\Big\{\int_1^{\pi\sqrt{(l^2+m^2)}} z^{-1}\,dz - \int_1^{\pi\sqrt{(l^2+m^2)}} z^{-1}J_0(z)\,dz + O(1)\Big\}. \tag{31}$$

The second integral is $O(1)$ and so

$$V_{l,n} = -\frac{RI}{4\pi}\{\log(l^2+m^2)+O(1)\}, \tag{32}$$

which is the kind of behaviour we might expect, knowing the logarithmic solution for the two-dimensional continuous case.

For the three-dimensional grid, the solution is

$$V_{l,m,n} = \frac{RI}{32\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{ilx+imy+inz}}{\sin^2\frac{1}{2}x+\sin^2\frac{1}{2}y+\sin^2\frac{1}{2}z}\,dxdydz.$$

We write

$$V_{l,m,n} = \frac{RI}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{ilx+imy+inz}}{x^2+y^2+z^2}\,dxdydz + \frac{RI}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{ilx+imy+inz}\times$$

$$\times\Big\{\frac{1}{4\sin^2\frac{1}{2}x+4\sin^2\frac{1}{2}y+4\sin^2\frac{1}{2}z}-\frac{1}{x^2+y^2+z^2}\Big\}\,dxdydz. \tag{33}$$

The second integrand is finite everywhere and so the integral is $O(1/lmn)$. This can be proved directly by integration by parts three times. So

$$V_{l,m,n} = \frac{RI}{8\pi^3}\Big\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{ilx+imy+inz}}{x^2+y^2+z^2}\,dxdydz + O\Big(\frac{1}{lmn}\Big)\Big\}$$

$$= \frac{RI}{8\pi^3}\Big\{\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} e^{ir\sqrt{(l^2+m^2+n^2)}\cos\theta}\sin\theta\,drd\theta d\phi + O\Big(\frac{1}{lmn}\Big)\Big\}, \tag{34}$$

where we have changed to polar coordinates, the domain of integration omitted making only a contribution of $O(1/lmn)$. So

$$V_{l,m,n} = \frac{RI}{4\pi^2}\left\{\int_0^\pi\int_0^\pi e^{ir\sqrt{(l^2+m^2+n^2)}\cos\theta}\sin\theta\,drd\theta + O\left(\frac{1}{lmn}\right)\right\}$$

$$= \frac{RI}{2\pi^2}\left\{\int_0^\pi \frac{\sin r\sqrt{(l^2+m^2+n^2)}}{r\sqrt{(l^2+m^2+n^2)}}\,dr + O\left(\frac{1}{lmn}\right)\right\}$$

$$= \frac{RI}{2\pi^2}\left\{\frac{1}{\sqrt{(l^2+m^2+n^2)}}\int_0^{\pi\sqrt{(l^2+m^2+n^2)}} \frac{\sin z}{z}\,dz + O\left(\frac{1}{lmn}\right)\right\}$$

$$= \frac{RI}{2\pi^2}\left\{\frac{1}{\sqrt{(l^2+m^2+n^2)}}\,\mathrm{Si}\{\pi\sqrt{(l^2+m^2+n^2)}\} + O\left(\frac{1}{lmn}\right)\right\}, \quad (35)$$

where [(1) 3]

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt. \quad (36)$$

Now

$$\mathrm{Si}(x) \approx \tfrac{1}{2}\pi - \frac{\cos x}{x} \quad \text{for } x \gg 1, \quad (37)$$

and so

$$V_{l,m,n} = \frac{RI}{4\pi}\left\{\frac{1}{\sqrt{(l^2+m^2+n^2)}} + O\left(\frac{1}{l^2+m^2+n^2}\right)\right\}, \quad (38)$$

which is the sort of asymptotic behaviour we might expect, knowing the solution in the three-dimensional continuous case.

6. Conclusion

The Green's functions and their asymptotic behaviour for Poisson's partial difference equation in two and three dimensions have been discussed. Non-isotropic grids can be treated in the same way and the method can be extended easily to triangular and hexagonal grids in two dimensions and probably to more complicated patterns in three dimensions.

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A TRANSFORM TECHNIQUE FOR BOUNDARY-VALUE PROBLEMS IN FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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[Received 3 August 1955]

1. Introduction

In a previous paper (**1**) I gave a transform technique which was suitable in obtaining the solutions to boundary-value problems in partial differential equations which contained Sturm–Liouville type of terms. Here I should like to develop a similar technique for boundary-value problems involving fourth-order partial differential equations.

In static and dynamic problems involving continuous media and in many other fields of engineering sciences partial differential equations containing fourth-order terms occur. The boundary-value problems for this type of equation become much more complex as compared with the ordinary wave equation or Laplace's equation. Moreover, the number of boundary conditions is increased, and these conditions are more complicated in nature.

The basic idea is to find a suitable kernel for an integral transform which will satisfy the given type of boundary conditions on one variable and reduce the dimension of the partial differential equation by one. A general method is given for this problem in which boundary values at each boundary are not coupled. This, of course, is the case in almost all technical problems. I deal only with the class of problems having boundaries in the finite range, and I restrict this study to partial differential equations which have no mixed derivatives or which can be reduced to this form. It is my hope to give a study of the other cases in a later paper.

2. The fourth-order self-adjoint system

A fourth-order, linear, homogeneous differential equation containing an arbitrary parameter λ has the general form

$$L_0 y+\lambda g(x)y(x) = 0, \qquad L_0 y = \sum_{i=0}^{4} a_i(x)y^{(i)}(x) \quad (a \leqslant x \leqslant b), \tag{1}$$

where y and x are the real dependent and independent variables, respectively, and

$$y^{(i)}(x) \equiv d^i y/dx^i, \qquad y^{(0)}(x) \equiv y(x), \qquad a_4 \neq 0.$$

The adjoint of (1) is defined by

$$M_0 y + \lambda g(x) y(x) = 0, \qquad M_0 y = \sum_{j=0}^{4} (-)^j (a_j y)^{(j)}. \tag{2}$$

Equation (1) is said to be *self-adjoint* if $L_0 y = M_0 y$. If (1) is self-adjoint, we can easily show that

$$a_4' = \tfrac{1}{2} a_3, \qquad -a_3'' + 2a_2' = 2a_1. \tag{3}$$

Any fourth-order, self-adjoint equation can always be transformed to

$$Lu + \lambda h(x) u = 0, \qquad Lu = u^{(\text{iv})} + (pu')' + qu \quad (a \leqslant x \leqslant b), \tag{4}$$

where

$$\begin{aligned} p(x) &= -\tfrac{3}{8}(b_3^2 + 4b_3') + b_2, \\ q(x) &= -\tfrac{1}{4}b_3''' + \tfrac{3}{16}b_3'^2 + \tfrac{3}{32}b_3^2 b_3' - \tfrac{3}{256}b_3^4 + \tfrac{1}{16}b_2(b_3^2 - 4b_3') - \tfrac{1}{4}b_3 b_1 + b_0, \\ y(x) &= u(x)\exp\Big[-\tfrac{1}{4}\int b_3(x)\,dx\Big], \qquad b_i = a_i/a_4, \end{aligned} \tag{5}$$

and primes represent differentiations. It is simpler to work with (4).

For a self-adjoint equation between Lu and Lv we have the Lagrange identity,

$$\left.\begin{aligned} vLu - uLv &= dP(u,v)/dx \\ P(u,v) &= p(u'v - uv') + u'v'' - u''v' + u'''v - uv''' \end{aligned}\right\}. \tag{6}$$

A two-point boundary-value problem having uncoupled end conditions consists in solving (4) under *four independent* end conditions specified at the end points $x = a$ and $x = b$ of an interval $a \leqslant x \leqslant b$, i.e.

$$\begin{aligned} U_i(u) &\equiv -\sum_{j=1}^{4} a_{ij} u^{(j-1)}(a) = 0, \\ \overline{U}_i(u) &\equiv \sum_{j=1}^{4} b_{ij} u^{(j-1)}(b) = 0 \quad (i = 1, 2), \end{aligned} \tag{7}$$

where a_{ij} and b_{ij} are constants such that U_i and $\overline{U}_i$ are linearly independent among themselves. This is true if, of the six 2×2 determinants contained in the matrix

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \tag{8}$$

at least one is not zero. The same must be true for (b_{ij}).

Let us select additional uncoupled end conditions U_3, U_4, $\overline{U}_3$, $\overline{U}_4$ such that U_j and $\overline{U}_j$ are linearly independent separately. To this end we

extend the range of subscript i in (7) to (1, 2, 3, 4), with the conditions that a_{ij} and b_{ij} for ($i = 3, 4$) are arbitrary, subject to the conditions that U_j and $\overline{U}_j$ are linearly independent. It is known (2) that the Lagrange's identity (6) can be integrated to give the Green's identity

$$\int_a^b (vLu - uLv)\, dx = \sum_{j=1}^{4} (U_j V_{5-j} + \overline{U}_j \overline{V}_{5-j}), \tag{9}$$

where V_j and $\overline{V}_j$ can be determined by comparing (9) with (6) and (7). It can be shown that, if the U_i, $\overline{U}_i$ are uncoupled, then the V_i, $\overline{V}_i$ are necessarily uncoupled.

Let the system adjoint to (4) and (7) be

$$Lv + \lambda h(x)v = 0, \qquad V_i(v) = 0, \qquad \overline{V}_i(v) = 0 \quad (a \leqslant x \leqslant b;\ i = 1, 2). \tag{10}$$

Then the system (4) and (7) will be *self-adjoint* if and only if

$$V_i(v) = \sum_{j=1}^{2} c_{ij}\, U_j(v), \qquad \overline{V}_i(v) = \sum_{j=1}^{2} \bar{c}_{ij}\, \overline{U}_j(v) \quad (i = 1, 2), \tag{11}$$

where c_{ij} and $\bar{c}_{ij}$ are constants. Given a_{ij} and b_{ij} we can determine c_{ij} and $\bar{c}_{ij}$ uniquely. Since the problem has partial symmetry with respect to the end conditions at $x = a$ and at $x = b$, it is sufficient to determine c_{ij}. The coefficients $\bar{c}_{ij}$ are obtained from c_{ij} by replacing a_{ij} by $-b_{ij}$ for ($i = 1, 2$), a_{ij} by b_{ij} for ($i = 3, 4$), and a by b respectively.

Let the six determinants of (8) be called d_{ij}, i.e.

$$\left.\begin{aligned} d_{12} &= a_{11}a_{22} - a_{21}a_{12}, & d_{23} &= a_{12}a_{23} - a_{22}a_{13} \\ d_{13} &= a_{11}a_{23} - a_{21}a_{13}, & d_{24} &= a_{12}a_{24} - a_{22}a_{14} \\ d_{14} &= a_{11}a_{24} - a_{21}a_{14}, & d_{34} &= a_{13}a_{24} - a_{23}a_{14} \end{aligned}\right\}. \tag{12}$$

The solutions for the c_{ij} are below.

(i) *Case* $d_{12} \neq 0$. Select $a_{3j} = a_{4j} = 0$ except $a_{33} = a_{44} = -1$; then

$$\begin{aligned} c_{11}d_{12} &= -p(a)a_{24} + a_{22}, & c_{21}d_{12} &= -p(a)a_{23} + a_{21}, \\ c_{12}d_{12} &= p(a)a_{14} - a_{12}, & c_{22}d_{12} &= p(a)a_{13} - a_{11}. \end{aligned} \tag{13}$$

(ii) *Case* $d_{13} \neq 0$. Select $a_{3j} = a_{4j} = 0$ except $a_{32} = a_{44} = -1$; then

$$\begin{aligned} c_{11}d_{13} &= a_{23}, & c_{21}d_{13} &= p(a)a_{23} - a_{21}, \\ c_{12}d_{13} &= -a_{13}, & c_{22}d_{13} &= -p(a)a_{13} + a_{11}. \end{aligned}$$

(iii) *Case* $d_{14} \neq 0$. Select $a_{3j} = a_{4j} = 0$, except $a_{32} = a_{43} = -1$; then

$$\begin{aligned} c_{11}d_{14} &= -a_{23}, & c_{21}d_{14} &= p(a)a_{24} - a_{22}, \\ c_{12}d_{14} &= a_{13}, & c_{22}d_{14} &= -p(a)a_{14} + a_{12}. \end{aligned}$$

(iv) *Case* $d_{23} \neq 0$. Select $a_{3j} = a_{4j} = 0$, except $a_{31} = a_{44} = -1$; then

$$c_{11} d_{23} = a_{24}, \qquad c_{21} d_{23} = -p(a)a_{23}+a_{21},$$
$$c_{12} d_{23} = -a_{14}, \qquad c_{22} d_{23} = p(a)a_{13}-a_{11}.$$

(v) *Case* $d_{24} \neq 0$. Select $a_{3j} = a_{4j} = 0$, except $a_{31} = a_{43} = -1$; then

$$c_{11} d_{24} = -a_{24}, \qquad c_{21} d_{24} = -p(a)a_{24}+a_{22},$$
$$c_{12} d_{24} = a_{14}, \qquad c_{22} d_{24} = p(a)a_{14}-a_{12}.$$

(vi) *Case* $d_{34} \neq 0$. Select $a_{3j} = a_{4j} = 0$ except $a_{31} = a_{42} = -1$; then

$$c_{11} d_{34} = a_{24}, \qquad c_{21} d_{34} = a_{23},$$
$$c_{12} d_{34} = -a_{14}, \qquad c_{22} d_{34} = -a_{13}.$$

By using the conditions of self-adjointness (11) we can prove† (5) the theorem:

Theorem 1. *A necessary and sufficient condition for the system* (4) *and* (7) *to be self-adjoint is that at $x = a$ and at $x = b$ we must have*

$$pd_{34}-d_{14}+d_{23} = 0. \tag{14}$$

3. T-transforms

Consider the self-adjoint system

$$Lu+\lambda u = 0, \qquad U_i(u) = 0, \qquad \overline{U}_i(u) = 0 \quad (i = 1, 2), \tag{15}$$

where Lu, U_i, and $\overline{U}_i$ are given by (4) and (7). Let $u_n(x) \equiv u(x, \lambda_n)$ be an *eigenfunction* of (15) corresponding to an *eigenvalue* λ_n.

It is known (3) that, if the system (15) is self-adjoint, then

(i) *there exist infinitely many eigenvalues and eigenfunctions which make a complete closed system*; (ii) *all eigenvalues are real*; (iii) *eigenfunctions are orthogonal*; (iv) *any function $f(x)$ satisfying usual conditions can be developed into a series of eigenfunctions of* (15) *in a unique way and this series converges uniformly and absolutely in the open interval $a < x < b$, i.e.*

$$f(x) = \sum_n A_n u_n(x), \qquad A_n = N_n^{-2} \int_a^b f(x)u_n(x)\, dx \tag{16}$$

since we have the orthogonality condition

$$\int_a^b u_n u_m\, dx = N_m^2 \delta_{mn}, \qquad \delta_{mn} = \begin{cases} 1 & (m = n), \\ 0 & (m \neq n). \end{cases} \tag{17}$$

† I am indebted to the referee for pointing out that the proof of this theorem was also given by H. T. Davis (4).

DEFINITION. *Let $f(x)$ be a real and absolutely integrable function of x, in the interval (a,b). We define the finite T-transform, $T\{f\}$, associated with the system* (15) *by*

$$f_T(\lambda_n) \equiv T\{f\} = N_n^{-2} \int_a^b f(x)u_n(x)\,dx. \tag{18}$$

The *inversion theorem* now follows from the second equation of (16)

$$f(x) = \sum_n f_T(\lambda_n)u_n(x), \tag{19}$$

where the summation is taken over the range of all eigenvalues of (15).

THEOREM 2. *If $f(x)$ is a continuously differentiable function of a real variable x up to and including the fourth order, then*

$$-T\{Lf\} = \lambda_n f_T(\lambda_n)+B\{f,u_n\}, \tag{20}$$

$$B\{f,u_n\} \equiv N_n^{-2}[V_1(f)U_4(u_n)+V_2(f)U_3(u_n)+\bar{V}_1(f)\bar{U}_4(u_n)+\bar{V}_2(f)\bar{U}_3(u_n)].$$

Proof. By Green's identity we have

$$-\int_a^b (f\,Lu_n-u_n\,Lf)\,dx = \sum_{j=1}^4 [U_j(u_n)V_{5-j}(f)+\bar{U}_j(u_n)\bar{V}_{5-j}(f)]. \tag{21}$$

We add to the integrand on the left $\lambda_n f u_n - \lambda_n f u_n$ and use the fact that u_n is an eigenfunction, i.e. it satisfies (15). Hence

$$-\int_a^b u_n\,Lf\,dx = \lambda_n \int_a^b u_n f\,dx + V_1(f)U_4(u_n)+V_2(f)U_3(u_n)+ \\ +\bar{V}_1(f)\bar{U}_4(u_n)+\bar{V}_2(f)\bar{U}_3(u_n). \tag{22}$$

In view of (18) this is the same as (20).

The significance of this theorem is that the T-transform can be used to exclude Lu from any partial differential equation which contains it. Moreover, any set of four boundary conditions $V_i = V_i(f)$, $\bar{V}_i = \bar{V}_i(f)$, $(i = 1, 2)$ can be taken into account automatically.

4. Solution of a class of boundary-value problems

Consider the class of boundary-value problems expressed by the *partial differential equation*

$$Lv = Mv,$$

$$Lv = v_{xxxx}+pv_{xx}+p_x v_x+qv, \qquad v = v(x,y),$$

$$Mv = \sum_{i=0}^m a_i(y)\,\partial^i v/\partial y^i, \qquad p = p(x,y), \qquad q = q(x,y),$$

$$(a_1 \leqslant x \leqslant b_1), \qquad (a_2 \leqslant y \leqslant b_2), \tag{23}$$

and the *boundary conditions*:

$$U_i[v(a_1,y)] = \phi_i(y), \qquad \overline{U}_i[v(b_1,y)] = \bar{\phi}_i(y) \qquad (i = 1, 2),$$

$$W_j(v) = \psi_j(x) \quad (j = 1, 2,..., m), \tag{24}$$

where U_i and $\overline{U}_i$ are of the form of (7) and W_j are any set of m linearly independent conditions specified at $y = a_2$ and $y = b_2$. Let $u_n(x, y)$, where y is treated as a parameter, be the eigenfunctions of the system (15). We now apply the T-transform to (23) with respect to x. Then

$$-\lambda_n v_T(\lambda_n, y) - B\{v, u_n\} = \sum_{i=0}^{m} a_i(y) d^i v_T(\lambda_n, y)/dy^i, \tag{25}$$

$$B\{v, u_n\}$$
$$= N_n^{-2} \sum_{j=1}^{2} [c_{1j}\phi_j U_4(u_n) + c_{2j}\phi_j U_3(u_n) + \bar{c}_{1j}\bar{\phi}_j \overline{U}_4(u_n) + \bar{c}_{2j}\bar{\phi}_j \overline{U}_3(u_n)].$$

Here $B\{v, u_n\}$ is a function of y alone. In arriving at (25) we used (11), (24), and the fact that u_n satisfies (15). The problem is now reduced to solving an ordinary linear differential equation (25) for the function $v_T(\lambda_n, y)$, under boundary conditions $T\{W_j\} = \psi_{jT}(\lambda_n)$ which are the transforms of $W_j(v) = \psi_j(x)$. To this end there is an extensive theory. After obtaining $v_T(\lambda_n, y)$ we take the inverse transform

$$v(x, y) = \sum_n v_T(\lambda_n, y) u_n(x, \lambda_n). \tag{26}$$

The present technique overcomes the following basic difficulty, namely, boundary conditions at both $x = a_1$, $x = b_1$, and $y = a_2$ and $y = b_2$ are automatically satisfied. This difficulty generally cannot be resolved by the technique of separation of variables or by well-known transforms such as those of Laplace, Fourier, etc. Hence boundary-value problems requiring two sets of boundary values at the boundaries can be conveniently solved.

5. Application

As an illustration, I solve a vibrating beam problem with time-dependent boundary conditions. This problem has application in ram-jet valves. As far as I know, this problem has not previously been solved.

PROBLEM. *To determine the deflexion $v(x, t)$ of a vibrating cantilever beam under given time-dependent bending moment M and shearing force Q at the built-in end $x = 0$, and time-dependent deflexion and slope at the free end $x = 1$. Let the initial conditions require that the deflexion v be prescribed at $t = 0$ and at a later time $t = t_1$.*

The differential equations (D.E.), the boundary conditions (B.C.), and the initial conditions (I.C.) are given by:

$$\text{(D.E.)} \quad \frac{\partial^4 v}{\partial x^4}+\frac{\partial^2 v}{\partial t^2}=0, \tag{27}$$

$$\text{(B.C.)} \quad \left.\begin{aligned} v(1,t) &= v_0(t), & v'(1,t) &= v_1(t) \\ v''(0,t) &= v_2(t), & v'''(0,t) &= v_3(t) \end{aligned}\right\}, \tag{28}$$

$$\text{(I.C.)} \quad v(x,0)=w_0(x), \qquad v(x,t_1)=w_1(x), \tag{29}$$

where $v'(1,t) = [\partial v/\partial x]_{x=1}$, etc. I have used

$$t=(D/\rho L^4)^{\frac{1}{2}}t_0, \qquad x=x_0/L, \qquad v=v_0/L$$

to transfer (27) from actual time t_0, length L, and displacement v_0 to non-dimensional time t, length 1, and displacement v. Here D and ρ are the flexural rigidity and mass density per unit length, respectively. We note that $M=-DL^{-1}v''$ and $Q=DL^{-2}v'''$. Comparing (27) and (28) with (23) and (24) we find that

$$p\equiv q\equiv 0, \qquad Mv=-\partial^2 v/\partial t^2, \qquad \phi_1\equiv v_2,$$
$$\phi_2\equiv v_3, \qquad \bar{\phi}_1\equiv v_0, \qquad \bar{\phi}_2\equiv v_1.$$

Hence $a_{ij}=b_{ij}=0$, except $a_{13}=a_{24}=-1$, $b_{11}=b_{22}=1$. We thus see from (12) that at $x=0$, $d_{34}=1$ and at $x=1$, $d_{12}=1$; all other $d_{ij}=0$. The condition of self-adjointness (14) is satisfied. From the list (13) of c_{ij} for the two cases $d_{34}\neq 0$ and $d_{12}\neq 0$ we find

$$-c_{11}=c_{22}=1, \qquad c_{12}=c_{21}=0 \quad \text{at } x=0,$$

and

$$-\bar{c}_{11}=\bar{c}_{22}=1, \qquad \bar{c}_{12}=\bar{c}_{21}=0 \quad \text{at } x=1.$$

Hence $B\{v,u_n\}$ given by the second of (25) takes the form

$$B\{v,u_n\} = F_n(t) = N_n^{-2}[-u_n'(0)v_2(t)+u_n(0)v_3(t)+u_n'''(1)v_0(t)-u_n''(1)v_1(t)], \tag{30}$$

where $u_n(x)$ and μ_n are the eigenfunctions and the eigenvalues of the system

$$\frac{d^4u_n}{dx^4}-\mu_n^4 u_n=0, \qquad \mu_n^4=-\lambda_n, \tag{31}$$

$$u_n(1)=u_n'(1)=u_n''(0)=u_n'''(0)=0,$$

which are respectively given by

$$\left.\begin{aligned} u_n(x) &= \frac{\cosh\mu_n(1-x)-\cos\mu_n(1-x)}{\cosh\mu_n+\cos\mu_n}-\frac{\sinh\mu_n(1-x)-\sin\mu_n(1-x)}{\sinh\mu_n+\sin\mu_n} \\ & \cosh\mu_n\cos\mu_n+1=0. \end{aligned}\right\} \tag{32}$$

We also find that

$$N_n^2=\tfrac{1}{4}[u_n^2(x)]_{x=0}. \tag{33}$$

Hence $F_n(t)$ is completely determined. We now apply the T-transform to (27). Hence

$$\frac{d^2 v_T}{dt^2}+\mu_n^4 v_T = F_n(t). \tag{34}$$

The solution of (34) satisfying (29) is

$$\begin{aligned} v_T(\mu_n,t) = (\sin\mu_n^2 t_1)^{-1}\Big[& w_{0T}(\mu_n)\sin\mu_n^2(t_1-t)+w_{1T}(\mu_n)\sin\mu_n^2 t- \\ & -(\mu_n^{-2}\sin\mu_n^2 t)\int\limits_0^{t_1} F_n(\tau)\sin\mu_n^2(t_1-\tau)\,d\tau+ \\ & +(\mu_n^{-2}\sin\mu_n^2 t_1)\int\limits_0^{t} F_n(\tau)\sin\mu_n^2(t-\tau)\,d\tau\Big], \end{aligned} \tag{35}$$

provided of course that $\sin\mu_n^2 t_1 \neq 0$. Here $w_{0T}(\mu_n)$ and $w_{1T}(\mu_n)$ are the T-transforms of $w_0(x)$ and $w_1(x)$ respectively. The solution is now completed if we use the inversion theorem (19).

$$v(x,t) = \sum_n v_T(\mu_n,t)u_n(x), \tag{36}$$

where the summation is extended over all positive roots of the second equation of (32).

It is important to note that this series will converge to $v(x,t)$ everywhere in the *open interval* $0 < x < 1$. However, at the end point $x = 0$, u'' and u''', and at $x = 1$, u and u' cannot be calculated by direct substitution of $x = 0$ and $x = 1$. In calculating these values we need to sum the series obtained and then substitute $x = 0^+$ and $x = 1^-$. This fact may also be expressed by limits of the form

$$v(1,t) = \lim_{x\to 1}\sum_n v_T(\mu_n,t)u_n(x), \quad ...,$$

where limit and summation are not interchangeable. Note that, when v satisfies the same conditions as u_n, then the convergence to boundary values will also be uniform.

It may be worth while to remark that by use of the Laplace-transform technique, which is used commonly in solving the time-dependent boundary-value problems, or by the technique of separation of variables, the solution of this problem would be extremely cumbersome if not impossible.

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FINITE GROUPS HAVING ISOMORPHIC IMAGES IN EVERY FINITE GROUP OF WHICH THEY ARE HOMOMORPHIC IMAGES

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1. THE problem of determining these groups was raised by Kertész (**1**), and is solved in this note. The solution is:

THEOREM 1. *The finite group G has the property that every finite group H of which G is a homomorphic image has a subgroup isomorphic to G if and only if it is either cyclic, or the direct product of two cyclic groups of which one has square-free order, but not both have twice odd order.*

The proof falls into two parts. We first use general arguments to prove that such a group must be abelian. The main stage here is the following:

THEOREM 2. *If R is a normal subgroup of a free group F, then $F/[R,R]$ has no elements of finite order except the identity.*

In the second stage of the proof the abelian groups are examined in turn.

2. An *abelian extension* of a group G is a group X with an abelian normal subgroup N such that $X/N \simeq G$; the extension *splits* if X has a subgroup Y such that $X = YN$ and $Y \cap N = 1$.

LEMMA 1. *Let F be a free group and R a normal subgroup of F. Then the extension $F/[R,R]$ of F/R splits if and only if every abelian extension of F/R splits.*

Since $F/[R,R]$ is an abelian extension of F/R, the 'if' is trivial. To prove the 'only if', assume that $F^*/[R,R]$ is a complement of $R/[R,R]$ in $F/[R,R]$, and that X is a group with an abelian normal subgroup N such that $X/N \simeq F/R$. Because F is free, there is a homomorphism α of F into X which induces this isomorphism between F/R and X/N. We put $Y = F^*\alpha$. Every co-set of N in X corresponds to a co-set of R in F, and this co-set contains an element of F^*. Thus every co-set of N in X contains an element of Y, which is to say that $X = YN$. But, if $g \in Y \cap N$, $g = f\alpha$ for some f in F^* because $g \in Y$, and $f \in R$ because $g \in N$. But $F^*/[R,R]$ is a complement of $R/[R,R]$, so that this implies $f \in [R,R]$. But α maps R into the abelian group N, so that $[R,R]$ is in

the kernel of α: that is $g = 1$, i.e. $Y \cap N = 1$. Thus Y is a complement of N, which proves the lemma.

COROLLARY 1. *If F, F' are free groups, with normal subgroups R, R' respectively, such that $G \simeq F/R \simeq F'/R'$, then the extensions $F/[R, R]$ and $F'/[R', R']$ of G either both split or both do not split.*

COROLLARY 2. *$F/[R, R]$ has no elements of finite order except the identity.*

Suppose on the contrary that the element f of F, not in $[R, R]$, is of finite order *modulo* $[R, R]$. Since $R/[R, R]$ is a free abelian group, $\{f, [R, R]\}/[R, R]$ is a complement to $R/[R, R]$ in $\{f, R\}/[R, R]$. But $\{f, R\}$ is a free group and $\{f, R\}/R$ is a cyclic group of finite order; and, if we choose S in the free cyclic group C such that C/S is of the same finite order, it is evident that $S/[S, S]$ has no complement in $C/[S, S]$. This contradiction with Corollary 1 establishes Corollary 2.

LEMMA 2. *A group G satisfying the hypothesis of Theorem 1 is abelian.*

Let F be a finitely-generated free group having a factor group F/R isomorphic to G. We put $X = F/[R, R]$, $N = R/[R, R]$. Let n be an integer such that $g^n = 1$ for all g in G (for instance, the order of G) and put $H = X/N^n$, where N^n is the group generated by the nth powers of elements in N. N is abelian by definition, and is finitely generated because it is a subgroup of finite index in a finitely-generated group. Thus H is finite, and it obviously has G as a homomorphic image. Hence H must have a subgroup isomorphic to G. We shall show that any element of H of order dividing n lies in N/N^n. For, if $g \in X$ and $g^n \in N^n$, we have evidently

$$g^n = a^n \quad (a \in N). \tag{1}$$

This clearly implies

$$a^{-n}g^{-1}a^n g = 1.$$

But a and $g^{-1}ag$ are both in N and therefore commute, so that this can be written

$$(a^{-1}g^{-1}ag)^n = 1.$$

By the second corollary to Lemma 1, X has no elements of finite order except identity, so that this implies $a^{-1}g^{-1}ag = 1$: that is, that a and g commute. Then (1) is equivalent to $(ga^{-1})^n = 1$, so that the same argument gives $g = a$. Thus $g \in N$, so that all elements of H of order dividing n are in N/N^n, as asserted. In particular, the subgroup of H isomorphic to G must be in N/N^n, which is abelian, and hence G is abelian.

3. Let us for brevity call a finite group G *good* if it satisfies the hypothesis of Theorem 1, and *bad* otherwise. The separation of the good abelian groups from the bad is facilitated by the following lemma, which I owe to Dr. P. M. Cohn.

Lemma 3. *If the group G is bad, there is a finite group H, with no subgroup isomorphic to G, having a minimal normal subgroup N in its Frattini subgroup such that $H/N \simeq G$.*

Since G is bad, there is a group H which exhibits its badness: that is, which has no subgroup isomorphic to G, but has a factor group H/N isomorphic to G. We suppose H chosen of order as small as possible. If M is a maximal subgroup of H not containing N, then $MN = H$, and so $M/M \cap N \simeq MN/H \simeq G$. Since M has no subgroup isomorphic to G, this contradicts the assumption that H was taken as small as possible. Thus N lies in all maximal subgroups of H, and hence in its Frattini subgroup. If N is not a minimal normal subgroup of H, let N_0 be a normal subgroup satisfying $N > N_0 > 1$. If H/N_0 has no subgroup isomorphic to G, it is a smaller group than H and exhibits the badness of G; if it has a subgroup H_0/N_0 isomorphic to G, then H_0 is a group smaller than H which exhibits the badness of G. In either case we have a contradiction, so that N is a minimal normal subgroup of H.

Corollary. *If G is abelian, then H is nilpotent and N is a subgroup of prime order in its centre.*

For, if G is abelian, the derived group $[H, H]$ is in N and hence in the Frattini subgroup. By a theorem of Wielandt's [cf. Zassenhaus (**2**) 113], H is nilpotent. Then H has no minimal subgroups except central subgroups of prime order.

We next show that we can confine our attention to primary groups.

Lemma 4. *The abelian group G is good if and only if all its Sylow subgroups are.*

Let all the Sylow subgroups of G be good. If H is a nilpotent group having G as a homomorphic image, each Sylow subgroup of H has the corresponding Sylow subgroup of G as a homomorphic image. Because these Sylow subgroups are good, H contains an isomorphic copy of each of them, and hence of G, because both G and H are the direct products of their Sylow subgroups. Thus no nilpotent group H can exhibit the badness of G. By the corollary to Lemma 3, G is not bad.

Conversely, if G is good and P is a Sylow subgroup of G, G is a direct product $P \times Q$. If H has P as a homomorphic image, then $H \times Q$ has G as a homomorphic image, and so has a subgroup isomorphic to G. The image of P in this isomorphism must lie in H because the elements of P and those of Q have co-prime orders.

LEMMA 5. *If the abelian p-group G is good, so is every direct factor of G.*

Suppose, on the contrary, that $G = A \times B$, with A bad and of order p^a. Its badness is exhibited by a group H of order p^{a+1}, which, having A as a homomorphic image but not as a subgroup, cannot be abelian. Because G is good, $H \times B$ has a subgroup G_0 isomorphic to G. Now G_0 is abelian, and H is not; so that the projection of G_0 into H cannot be the whole of H. Since G_0 is of index p, this implies that $G_0 = A_0 \times B$, for some A_0 in H. But $A_0 \times B \simeq A \times B$ implies $A_0 \simeq A$, contrary to the fact that H exhibits the badness of A. This contradiction establishes the lemma.

LEMMA 6. *An abelian p-group G of type (i,j) $(i \geqslant j > 1)$ is bad.*

The group H generated by a, b, c subject to the relations

$$a^{p^i} = b^{p^j} = c^p = 1, \qquad [a,b] = c, \qquad [a,c] = [b,c] = 1$$

has G as a homomorphic image. But a subgroup isomorphic to G, being of index p, would have to contain the Frattini subgroup, an abelian group of type $(i-1, j-1, 1)$. This is impossible, which proves the lemma.

LEMMA 7. *An abelian p-group G of type $(i,1,1)$ is bad.*

If $p = 2$, the quaternion group shows that a group of type $(1,1)$ is bad. Thus we may assume that p is odd. Then the group H generated by a, b, c, d subject to the relations

$$a^p = d, \qquad b^p = c^p = d^p = 1, \qquad [b,c] = d, \qquad [a,b] = [a,c] = 1$$

has G as a homomorphic image. A subgroup isomorphic to G can contain no element outside the subgroup $\{a^p, b, c, d\}$ since these elements have order p^{i+1}. But neither can it coincide with this subgroup, which is not abelian. Thus G is bad.

Lemmas 4 to 7, together with the fact, mentioned in the proof of Lemma 7, that an abelian 2-group of type $(1,1)$ is bad, show that no abelian group not in the list in Theorem 1 can be good. To complete the proof of the theorem all that is necessary is the following:

LEMMA 8. *Except in the case $p = 2$, $i = 1$, an abelian p-group G of type $(i, 1)$ is good.*

For $i = 1$, p odd, this is practically a restatement of the well-known theorem of Burnside's that a non-cyclic p-group contains more than one subgroup of order p if p is odd [cf. Zassenhaus (**2**) 118].

By Lemma 3, it is sufficient to show that, if a group H of order p^{i+2} has a homomorphism onto G, then it has a subgroup isomorphic to G.

We can take H to be generated by elements a, b, c subject to relations of the form

$$a^{p^i} = c^x, \qquad b^p = c^y, \qquad [a,b] = c^z, \qquad c^p = 1, \qquad [a,c] = [b,c] = 1.$$

Even if $p = 2$, H is not a generalized quaternion group if $i \geqslant 2$, since its commutator subgroup has index p^{i+1}. By the theorem quoted above, H has an element, b' say, of order p not in the cyclic group $\{c\}$. Then as subgroup isomorphic to G we may take $\{a^p, b'\}$ or $\{a, c\}$ according as $x \neq 0$ or $x = 0$.

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AN INTEGRAL OF PERRON'S TYPE DEFINED WITH THE HELP OF TRIGONOMETRIC SERIES

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1. Introduction

A TRIGONOMETRIC series of the form

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}A_n(x), \qquad A_n(x) = a_n\cos nx+b_n\sin nx \quad (n = 1, 2,...) \tag{1}$$

can converge everywhere to the periodic function $f(x)$ without $f(x)$ being integrable in Lebesgue's sense or even in Denjoy's sense. If it were integrable, we should like to say that

$$\tfrac{1}{2}a_0x-\sum_{n=1}^{\infty}\frac{B_n(x)}{n}, \qquad B_n(x) = b_n\cos nx-a_n\sin nx \tag{2}$$

in some sense represents the primitive function of $f(x)$. Difficulties arise in the first place because the series (2) need not converge for all x, so that the sum function $F(x)$ (which is defined and finite almost everywhere) need not be continuous.

Two distinct methods of attacking this problem have been developed. (i) Integrals of Perron's type have been defined in which the primitive function need exist only at points of a basis B which is of full measure in the interval of integration: the (T)-integral of Marcinkiewicz–Zygmund (**5**), and the (SCP)-integral of Burkill (**1**) are of this type. (ii) Instead of defining a first-order integral, we might define a double-integration process to obtain a second primitive. If the series (1) converges on a set of positive measure, the coefficients a_n, b_n must tend to zero, and the series

$$\tfrac{1}{4}a_0x^2-\sum_{n=1}^{\infty}\frac{A_n(x)}{n^2} \tag{3}$$

converges everywhere to a continuous smooth function $\Phi(x)$. A second primitive for $f(x)$ which differs from $\Phi(x)$ by only a linear function can be defined by such a double integration process. The 'Totalisation à deux degrés' of Denjoy [see (**2**)], and the (P^2)-integral of James (**4**) are of this type. Using any of the four integrals SCP, T, D^2, P^2 we can show that an everywhere-convergent trigonometric series is the Fourier series of its sum.

Further difficulties start to arise if we relax the condition that (1) converges, and replace it by the condition that it is summable in some sense. Actually the SCP-integral suffices to integrate the $(R, 2)$ sum of (1) provided that the coefficients satisfy

$$a_n = o(n^\gamma), \qquad b_n = o(n^\gamma) \quad \text{for some } \gamma < \tfrac{1}{2} \tag{4}$$

since this condition makes the Riesz–Fischer theorem applicable to the integrated series (2). However, the integrals mentioned do not seem to be sufficiently general to integrate $f(x)$ when it is only known to be the Abel sum of (1).

In the present paper we use some of the known properties of trigonometric series to define an integral. This integral will deal with the sum of a series (1) by any method of summation which implies Abel summability; in particular, it will apply when (1) is summable (C, k) $(k > -1)$, or $(R, 2)$, or $(R, 1)$.

By the classical Riemann theory, a trigonometric series (1) which converges to zero everywhere must have all its coefficients zero. However it is easy to show that

$$\lim_{r \to 1-0} \sum_{n=1}^{\infty} n r^n \sin n\theta = 0 \quad \text{for all } \theta.$$

Thus when dealing with series (1) which are Abel-summable we shall have to impose conditions on the coefficients which are sufficient to imply uniqueness. The simplest condition of this kind is

$$a_n = o(n), \qquad b_n = o(n). \tag{5}$$

That this condition implies uniqueness for a trigonometric series which is everywhere Abel summable was first proved by Verblunsky (9).

The integral which I define will be called the *Abel–Perron* or the (AP)-integral. It will combine some of the ideas of the (SCP)-integrals and (P^2)-integrals. For any function $f(x)$ which is (AP)-integrable over (a, b) we obtain

(i) a value for $\int_a^b f(x)\,dx$, but not necessarily for $\int_a^y f(x)\,dx$ for any y in (a, b);

(ii) a second primitive $\Phi(x)$ defined for $a \leqslant x \leqslant b$.

We shall see that the (AP)-integral includes the ordinary Perron integral: also it is 'non-absolute' in the sense that a positive (AP)-integrable function is Lebesgue-integrable to the same value.

It remains an unsolved problem whether or not the Abel summability of (1) everywhere, together with the uniqueness condition (5), implies

that the integrated series (2) is Abel-summable in a set of full measure. For this reason we cannot obtain a first-order primitive defined almost everywhere; hence a function which is (AP)-integrable over (a,b) cannot be shown to be integrable over (a_1, b_1) for any values of a_1, b_1 satisfying $a < a_1 < b_1 < b$.

The usual properties of an integral which do not depend on integrability over a sub-interval are shown to have their analogues for the (AP)-integral.

2. Definitions and preliminary results

For any function $F(x)$ defined in an interval, write

$$\Delta^2 F(x,h) = F(x+h)+F(x-h)-2F(x).$$

$F(x)$ is said to be *smooth* at x_0 if

$$\lim_{h\to 0+} h^{-1}\Delta^2 F(x_0,h) = 0.$$

The generalized second derivates are defined by

$$\bar{D}^2 F(x) = \limsup_{h\to 0} h^{-2}\Delta^2 F(x,h),$$

$$\underline{D}^2 F(x) = \liminf_{h\to 0} h^{-2}\Delta^2 F(x,h).$$

When these two derivates are equal at a point x_0, $F(x)$ is said to have a *generalized second derivative* there, denoted by

$$\bar{D}^2 F(x_0) = \underline{D}^2 F(x_0) = D^2 F(x_0).$$

A function $\phi(x)$ which at every point of the interval $a \leqslant x \leqslant b$ has a unique value (the values $+\infty$ and $-\infty$ are allowed) will be said to *possess the property* R^* in that interval if, given any perfect set P contained in (a,b), there is a non-void portion of P on which $\phi(x)$ is upper semi-continuous. The property R_* is obtained by replacing 'upper' by 'lower' in the last sentence: $\phi(x)$ will be said to have the property R if it has both the properties R^* and R_*.

We need some results about convex functions.

Lemma 1. *Let $F(x)$ be defined and finite in $a < x < b$ and upper semi-continuous in that interval. Then, if $\bar{D}^2 F(x) \geqslant 0$ in (a,b), $F(x)$ is continuous and convex there.*

This is a special case of Lemma 14 of (**9**).

Theorem 1. *Let $F(x)$ be defined and finite in $a \leqslant x \leqslant b$, be approximately continuous, and possess the property R^* in that interval. Then, if $\bar{D}^2 F(x) \geqslant 0$ in (a,b), $F(x)$ is continuous and convex.*

Proof. Let E be the set of points ξ of $[a,b]$ such that $F(x)$ is not upper semi-continuous in any open interval containing ξ. By the definition of the property R^*, E is a nowhere-dense closed set. If σ is a contiguous open interval of $(a,b)-E$, then $F(x)$ is upper semi-continuous in σ. By Lemma 1, $F(x)$ is continuous and convex in σ. Suppose, if possible, that there is an isolated point ξ_0 of E. Then

$$\lim_{x\to\xi_0-0} F(x), \qquad \lim_{x\to\xi_0+0} F(x)$$

both exist. Since $F(x)$ is approximately continuous at ξ_0, these limits must be equal to $F(\xi_0)$, and $F(x)$ is continuous at ξ_0. This contradicts the definition of E; hence E is a perfect set.

Suppose now that E is not void. Let F be the set of points ξ which are not contained in any open interval I such that $F(x)$ is upper semi-continuous on $E\cap I$. Then F is a closed subset of E which is nowhere dense on E by the definition of the property R^*. Let s be any closed interval contained in $(a,b)-F$ and such that $E\cap s$ is not void. Now $F(x)$ is convex and continuous in the contiguous intervals of $s-E$, and therefore attains its maximum for such an interval at an end-point: that is, at a point of $E\cap s$. It follows that $F(x)$ is upper semi-continuous in s. Since $E\cap s$ was not void, this again contradicts the definition of E. Thus E is void, $F(x)$ is upper semi-continuous in (a,b), and the theorem follows from Lemma 1.

THEOREM 2. *Let $g(x)$ be continuous and convex in an interval (a,c) and let $a<\alpha<\gamma<c$. Let E_k be the set of points in (α,γ) for which $\overline{D}^2g(x)\geqslant k$. Then*

$$\frac{g(c)-g(\gamma)}{c-\gamma}-\frac{g(\alpha)-g(a)}{\alpha-a}\geqslant \tfrac{1}{4}k|E_k|,$$

where $|E_k|$ denotes the outer Lebesgue measure.

COROLLARY. *Let $g(x)$ be continuous and convex in (a,c). Then $\overline{D}^2g(x)<\infty$ almost everywhere in (a,c).*

This is Theorem 4 of (**3**).

We now consider Lebesgue-integrable functions $\Lambda(x)$ of period 2π. Let

$$\Lambda(r,x)=\frac{1}{\pi}\int_0^{2\pi}\Lambda(x+t)\frac{1-r^2}{1+r^2-2r\cos t}\,dt \tag{6}$$

for $0\leqslant r<1$. Now it is well known that we can write

$$\Lambda(r,x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)r^n, \tag{7}$$

where a_n, b_n are the Fourier coefficients of $\Lambda(x)$. A periodic function $\Lambda(x)$ will be said to be *Abel-continuous* or $\mathscr{A}$-continuous at x_0 if it is Lebesgue-integrable and

$$\lim_{r\to 1-0} \Lambda(r, x_0) = \Lambda(x_0).$$

We also need a more general second derivative for $\Lambda(x)$. Suppose $\Lambda(x)$ integrable and periodic of period 2π and $\Lambda(r,x)$ defined by (6). Then $\partial^2\Lambda(r,x)/\partial x^2$ exists and is continuous for $0 \leqslant r < 1$. Define

$$A\underline{D}^2\Lambda(x) = \liminf_{r\to 1-0} \frac{\partial^2}{\partial x^2}\Lambda(r,x),$$

$$A\overline{D}^2\Lambda(x) = \limsup_{r\to 1-0} \frac{\partial^2}{\partial x^2}\Lambda(r,x). \tag{8}$$

If $A\overline{D}^2\Lambda(x) = A\underline{D}^2\Lambda(x)$, we say that the *generalized Abel second derivative* exists, and write $AD^2\Lambda(x)$ for their common value.

Suppose now that $\quad \Phi(x) = \frac{1}{2}Kx^2 + \Lambda(x),$

where K is a finite constant and $\Lambda(x)$ is integrable and periodic with period 2π. We then define

$$A\underline{D}^2\Phi(x) = K + A\underline{D}^2\Lambda(x), \qquad A\overline{D}^2\Phi(x) = K + A\overline{D}^2\Lambda(x), \tag{9}$$

and $$AD^2\Phi(x) = K + AD^2\Lambda(x)$$

when the latter exists. Thus we have defined Abel second derivates for a somewhat wider class of functions. There are relations between the (AD^2)-derivates and the ordinary (D^2)-derivates which will be important for us. These are contained in

THEOREM 3. *Given a periodic integrable $\Lambda(x)$ such that x_0 is a point where $\Lambda(x)$ is $\mathscr{A}$-continuous; then*

(i) $\quad \overline{D}^2\Lambda(x_0) \geqslant A\underline{D}^2\Lambda(x_0), \qquad \underline{D}^2\Lambda(x_0) \leqslant A\overline{D}^2\Lambda(x_0)$;

(ii) *there exists an absolute constant σ $(1 < \sigma < \frac{11}{10})$ such that, if*

$$|\overline{D}^2\Lambda(x_0)| \leqslant M, \qquad |\underline{D}^2\Lambda(x_0)| \leqslant M$$

for a finite constant M, then

$$|A\overline{D}^2\Lambda(x_0)| \leqslant \sigma M, \qquad |A\underline{D}^2\Lambda(x_0)| \leqslant \sigma M.$$

Proof (i) These are simply the Rajchman inequalities translated into our notation. For a proof see, for example, (**12**) [p. 295]. (ii) This result follows easily from the calculations of (**6**) [p. 272].

THEOREM 4. *If $\Lambda(x)$ satisfies the conditions of Theorem 3, and x_0 is such that $D^2\Lambda(x_0)$ exists, then $AD^2\Lambda(x_0)$ exists and has the same value.*

The result, when $\Lambda(x)$ is continuous at x_0, is due to Fatou. For a proof of the result stated see Theorem 3, Corollary 2, of (**8**).

THEOREM 5. *Suppose $\Lambda(x)$ an integrable function of period 2π which is everywhere $\mathscr{A}$-continuous, and $\Lambda(r,x)$ defined by* (6). *Then at points ξ where $\Lambda(x)$ is smooth, we have*

$$\lim_{r\to 1-0}\left[(1-r)\frac{\partial^2}{\partial x^2}\Lambda(r,x)\right] = 0.$$

This is a special case of Lemma 19 of (**9**) translated into the notation of this paper.

3. Definition of the (AP)-integral over $(0, 2\pi)$

Suppose $f(x)$ defined p.p. in $(0, 2\pi)$: let it be defined p.p. for all real x by putting $f(x+2\pi) = f(x)$.

The finite constant M and the real function $\Phi(x)$ will be said to form an *AP upper approximating pair* if

(i) $\Lambda(x) = \Phi(x)-\dfrac{M}{4\pi}x^2$ is periodic with period 2π,

(ii) $\Lambda(x)$ is Lebesgue-integrable and $\mathscr{A}$-continuous for all x,

(iii) $\Lambda(x)$ is approximately continuous and has the property R^*,

(iv) $\Phi(-2\pi) = \Phi(2\pi) = 0$,

(v) $A\underline{D}^2\Phi(x) \geqslant f(x)$ p.p.; and $A\underline{D}^2\Phi(x) > -\infty$ except (possibly) in an enumerable set E,

(vi) if $\Lambda(r,x)$ is defined by (6) at all points of E, we have

$$\lim_{r\to 1-0}\left[(1-r)\frac{\partial^2}{\partial x^2}\Lambda(r,x)\right] = 0.$$

[Note that condition (vi) is weaker than the requirement that $\Phi(x)$ be smooth at points of E: this follows from Theorem 5.]

A *lower approximating pair* $\{m, \phi(x)\}$ is defined by making the obvious symmetric changes in the above definition. Then the function $f(x)$ is said to be *(AP)-integrable* over $(0, 2\pi)$ if and only if

$$\inf M = \sup m = I, \quad \text{say}, \tag{10}$$

where the bounds are taken over the class of all approximating pairs. Then we write

$$AP\text{-}\int_0^{2\pi} f(x)\,dx = I.$$

We must now show that this is a proper definition. The first step is to show that the points of the exceptional set E allowed by condition (v) above are not important. For this we need

LEMMA 2. *Given $\epsilon > 0$, and x_0 a point of $(0, 2\pi)$, there exists an upper approximating pair $\{Q, F(x)\}$ for the function $t(x) \equiv 0$ such that*

(i) *$F(x)$ is continuous for all x,*

(ii) *$A\underline{D}^2F(x) \geqslant 0$ for all x,*

(iii) *$A\underline{D}^2F(x_0) = +\infty$,*

and
$$\lim_{r\to 1-0}\left[(1-r)\frac{\partial^2}{\partial x^2}\Psi(r, x_0)\right] > 0,$$
where
$$\Psi(x) = F(x) - \tfrac{1}{4}\pi^{-1}Qx^2,$$
and $\Psi(r, x)$ is defined by (6),

(iv) *$0 < Q < \epsilon$, and $|F(x)| < \epsilon$ for $-2\pi \leqslant x \leqslant 2\pi$.*

Proof. Let
$$f(r, x) = \tfrac{1}{60}\epsilon\left[\tfrac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(x - x_0)\right].$$
Then, if $|x - x_0| \neq 2\pi m$ for any integer m,
$$\lim_{r\to 1-0} f(r, x) = 0.$$
But
$$\lim_{r\to 1-0} f(r, x_0) = +\infty,$$
and
$$\lim_{r\to 1-0}[(1-r)f(r, x_0)] = \tfrac{1}{60}\epsilon > 0.$$
Put
$$\Psi(x) = -\tfrac{1}{60}\epsilon\sum_{1}^{\infty}\frac{\cos n(x - x_0)}{n^2} + \lambda,$$
where λ is a positive constant chosen so that
$$\Psi(-2\pi) = \Psi(2\pi) = -\tfrac{1}{60}\pi^2\epsilon.$$
The series for $\Psi(x)$ is uniformly and absolutely convergent to a continuous function, and it is easy to see that
$$|\Psi(x)| < \tfrac{1}{2}\epsilon \quad \text{for all } x.$$
Now put $Q = \frac{1}{60}\pi\epsilon$, and $F(x) = \Psi(x) + \frac{1}{4}\pi^{-1}Qx^2$. The conditions (i)–(iv) are all satisfied by the pair $\{Q, F(x)\}$.

Remark. This lemma has some interest in itself. It can be used to simplify the methods of (**10**), for it eliminates at an early stage the points where
$$\liminf_{r\to 1-0}\sum r^n A_n(x) = -\infty, \qquad \lim_{r\to 1-0}\left[(1-r)\sum r^n A_n(x)\right] = 0,$$
which are the source of much complication in the author's proof.

LEMMA 3. *Suppose that $\{M, \Phi(x)\}$ is an upper approximating pair for $f(x)$ on $(0, 2\pi)$ and $\epsilon > 0$. Then there exists an upper approximating pair $\{M_1, \Phi_1(x)\}$ such that*

$$0 < M_1 - M < \epsilon, \qquad |\Phi(x) - \Phi_1(x)| < \epsilon \quad \text{for } -2\pi \leqslant x \leqslant 2\pi$$

and $A\underline{D}^2\Phi_1(x) > -\infty$ for all x.

Proof. Let E be the set of points where

$$A\underline{D}^2\Phi(x) = -\infty.$$

The set E is at most enumerable, and, by (vi), for $x \in E$,

$$\lim_{r \to 1-0}\left[(1-r)\frac{\partial^2}{\partial x^2}\Lambda(r, x)\right] = 0.$$

Suppose that $x_1, x_2, \ldots$ are the points of E in the interval $(0, 2\pi)$. Let $\{\epsilon_k\}$ be a sequence of positive numbers with

$$\sum_{k=1}^{\infty} \epsilon_k < \epsilon.$$

Let $\{Q_k, F_k(x)\}$ be the pair defined by Lemma 2 with ϵ, x_0 replaced by ϵ_k, x_k. Put

$$F(x) = \sum_{k=1}^{\infty} F_k(x), \qquad Q = \sum_{k=1}^{\infty} Q_k.$$

Then $F(x)$ is the sum of a uniformly convergent series of continuous functions. Hence it is continuous and

$$\Psi(x) = F(x) - \tfrac{1}{4}\pi^{-1}Qx^2$$

is periodic and Lebesgue-integrable. Further

$$\lim_{r \to 1-0}\left[(1-r)\frac{\partial^2}{\partial x^2}\Psi(r, x)\right] > 0$$

for x in E, and

$$A\underline{D}^2F(x) \geqslant 0$$

for all x.

Let
$$\Phi_1(x) = \Phi(x) + F(x), \qquad M_1 = M + Q.$$

Then
$$A\underline{D}^2\Phi_1(x) \geqslant A\underline{D}^2\Phi(x) + A\underline{D}^2F(x) \geqslant f(x) \quad \text{p.p.}$$

Also at points of E,

$$\lim\left[(1-r)\frac{\partial^2}{\partial x^2}\Lambda_1(r, x)\right]$$
$$= \lim\left[(1-r)\frac{\partial^2}{\partial x^2}\Psi(r, x)\right] + \lim\left[(1-r)\frac{\partial^2}{\partial x^2}\Lambda(r, x)\right] > 0;$$

so that we have
$$A\underline{D}^2\Phi_1(x) = +\infty.$$

Thus the pair $\{M_1, \Phi_1(x)\}$ satisfies the required conditions.

The next step is to show that the set of measure zero where

$$f(x) > A\underline{D}^2\Phi(x) > -\infty$$

is not important. For this we need

LEMMA 4. *Suppose that E is a set of measure zero contained in $(0, 2\pi)$, and $\epsilon > 0$: then there exists an upper approximating pair $\{Q, F(x)\}$ such that*

(i) *$F(x)$ is continuous and smooth for all x,*

(ii) *$A\underline{D}^2F(x) \geqslant 0$ for all x,*

(iii) *$A\underline{D}^2F(x) = +\infty$ for x in E,*

(iv) *$0 < Q < \epsilon$ and $|F(x)| < \epsilon$ for $-2\pi \leqslant x \leqslant 2\pi$.*

Proof. Let A be the set of points obtained from E by periodicity: that is, ξ is in A if $\xi+2r\pi$ is in E for some integer r. Given n, there is an open set H_n, containing E, of measure less than $\epsilon/30.4^n$. Let O_n be the set of points obtained from H_n by periodicity.

Define
$$\nu_n(x) = \begin{cases} 2^n & \text{for } x \text{ in } O_n, \\ 0 & \text{elsewhere}, \end{cases}$$

and let
$$\mu_n(x) = \int\limits_{-2\pi}^{x} \nu_n(t)\,dt.$$

Then $\mu_n(x)$ is continuous, increasing and $\mu_n(2\pi) < \epsilon/15.2^n$. Write

$$\mu(x) = \sum_1^\infty \mu_n(x), \qquad J(x) = \int\limits_{-2\pi}^{x} \mu(t)\,dt.$$

Then $J(x)$ is a convex smooth function, increasing in $(-2\pi, 2\pi)$, with $J(-2\pi) = 0$, $J(2\pi) < \epsilon$.

Write
$$F(x) = J(x) - \frac{x+2\pi}{4\pi} J(2\pi).$$

Then $F(x)$ is convex,
$$F(-2\pi) = F(2\pi) = 0,$$

and $|F(x)| < \epsilon$ in $(-2\pi, 2\pi)$. If $\Lambda(x) = F(x) - \frac{1}{4}\pi^{-1}Qx^2$, where $Q = \mu(0)$, it can be shown that $\Lambda(x)$ is periodic in $(-2\pi, 2\pi)$. Further, for x in E,
$$\underline{D}^2F(x) = +\infty, \qquad A\underline{D}^2F(x) = +\infty$$

by Theorem 4. Finally $A\underline{D}^2F(x) \geqslant 0$ for all x since it is differentiable and $\underline{D}\mu(x) \geqslant 0$ at all points.

LEMMA 5. *Suppose that $\{M, \Phi(x)\}$ is an upper approximating pair for $f(x)$ and $\epsilon > 0$. Then there exists an upper approximating pair $\{M_2, \Phi_2(x)\}$ such that*

$$0 < M_2 - M < \epsilon, \qquad |\Phi(x) - \Phi_2(x)| < \epsilon \quad \textit{for } -2\pi \leqslant x \leqslant 2\pi;$$
$$A\underline{D}^2\Phi_2(x) \geqslant f(x), \qquad A\underline{D}^2\Phi_2(x) > -\infty \quad \textit{for all } x.$$

Proof. Let $\{M_1, \Phi_1(x)\}$ be a pair satisfying the conditions of Lemma 3 with ϵ replaced by $\frac{1}{2}\epsilon$. Now suppose E the set of measure zero where $A\underline{D}^2\Phi_1(x) < f(x)$. Apply Lemma 4 to obtain a pair $\{Q, F(x)\}$ such that

$$A\underline{D}^2F(x) \geqslant 0 \quad \text{for all } x, \qquad A\underline{D}^2F(x) = +\infty \quad \text{in } E,$$

$$0 < Q < \tfrac{1}{2}\epsilon \quad \text{and} \quad |F(x)| < \tfrac{1}{2}\epsilon \quad \text{for } -2\pi \leqslant x \leqslant 2\pi.$$

Now put $$\Phi_2(x) = \Phi_1(x)+F(x), \qquad M_2 = M_1+Q.$$

It is easy to see that the pair $\{M_2, \Phi_2(x)\}$ satisfies the required conditions.

LEMMA 6. *Suppose that $\{M, \Phi(x)\}$, $\{m, \phi(x)\}$ are upper and lower AP approximating pairs for a given function $f(x)$. Then $M \geqslant m$, and $[\Phi(x)-\phi(x)]$ is convex for $-2\pi < x < 2\pi$.*

Proof. We may assume that the approximating pairs satisfy conditions (i)–(iv) of their definition and that, for all x,

$$A\underline{D}^2\Phi(x) \geqslant f(x) \geqslant A\bar{D}^2\phi(x), \qquad A\underline{D}^2\Phi(x) > -\infty,$$
$$A\bar{D}^2\phi(x) < +\infty$$

since, by Lemma 5, $\{M, \Phi(x)\}$, $\{m, \phi(x)\}$ are the uniform limits of pairs with this property. Thus for all x,

$$A\underline{D}^2[\Phi(x)-\phi(x)] \geqslant A\underline{D}^2\Phi(x)-A\bar{D}^2\phi(x) \geqslant 0.$$

Hence $$A\underline{D}^2[\Lambda(x)-\lambda(x)] \geqslant -\frac{1}{2\pi}(M-m);$$

and so, by Theorem 3,

$$\bar{D}^2[\Lambda(x)-\lambda(x)] \geqslant -\frac{1}{2\pi}(M-m)$$

and $$\bar{D}^2[\Phi(x)-\phi(x)] = \frac{1}{2\pi}(M-m)+\bar{D}^2[\Lambda(x)-\lambda(x)] \geqslant 0.$$

Similarly we have $\bar{D}^2[\Phi(x)-\phi(x)] > -\infty$ for all x. Since $[\Phi(x)-\phi(x)]$ is approximately continuous and has the property R^*, we may apply Theorem 1. This shows that $[\Phi(x)-\phi(x)]$ is convex. Hence $\Phi(0) \leqslant \phi(0)$. But $$\Phi(0) = \Lambda(0) = \Lambda(-2\pi) = -\pi M; \qquad \phi(0) = -\pi m.$$

Thus we have $M \geqslant m$.

This lemma shows that the definition of the integral (10) is a proper one. We now obtain a second-order primitive for an (AP)-integrable function.

LEMMA 7. *Suppose that $f(x)$ is (AP)-integrable over $(0, 2\pi)$ and $\epsilon > 0$. Then there exist approximating pairs $\{M_1, \Phi_1(x)\}$, $\{m_1, \phi_1(x)\}$ such that $0 \leqslant \phi_1(x)-\Phi_1(x) < \epsilon$ for $-2\pi \leqslant x \leqslant 2\pi$. Further there exists a second primitive function $F(x)$ such that $\Phi(x) \leqslant F(x) \leqslant \phi(x)$ $(-2\pi \leqslant x \leqslant 2\pi)$ for any approximating pairs.*

Proof. By (10), there exist pairs $\{M_1, \Phi_1(x)\}$, $\{m_1, \phi_1(x)\}$ with

$$0 \leqslant M_1 - m_1 < \epsilon/2\pi.$$

Now $$\Phi_1(-2\pi) = \Phi_1(2\pi) = \phi_1(-2\pi) = \phi_1(2\pi) = 0,$$

and therefore $$\phi_1(0) - \Phi_1(0) = \pi(M_1 - m_1).$$

Since $[\Phi_1(x) - \phi_1(x)]$ is convex, we then have

$$0 \leqslant \phi_1(x) - \Phi_1(x) \leqslant 2\pi(M_1 - m_1) < \epsilon \quad \text{for } -2\pi \leqslant x \leqslant 2\pi.$$

For x in $(-2\pi, 2\pi)$, write $F(x) = \sup \Phi(x)$, where the supremum is taken over all upper approximating pairs. For a fixed $\phi(x)$, $[\Phi(x) - \phi(x)]$ is convex for all $\Phi(x)$; hence $\Phi(x) \leqslant \phi(x)$ for all $\Phi(x)$, and so $F(x) \leqslant \phi(x)$. Since this is true for all $\phi(x)$, we have the inequality

$$\Phi(x) \leqslant F(x) \leqslant \phi(x)$$

in $(-2\pi, 2\pi)$.

4. Properties of the (AP)-integral

In the present section I assume throughout that $f(x)$ is defined p.p. on $(0, 2\pi)$ and is (AP)-integrable. Let $F(x)$ denote the second primitive of $f(x)$, defined by Lemma 6; then $F(-2\pi) = 0 = F(2\pi)$. Write

$$I = AP\text{-}\int_0^{2\pi} f(x)\,dx, \qquad L(x) = F(x) - \frac{I}{4\pi}x^2.$$

By Lemma 6, $L(x)$ is the uniform limit of periodic functions which are approximately continuous, Lebesgue-integrable, and $\mathscr{A}$-continuous. Hence

(i) $L(x)$ is periodic, and $F(0) = -\pi I$;
(ii) $L(x)$ is Lebesgue integrable and $\mathscr{A}$-continuous for all x;
(iii) $F(x)$ is approximately continuous.

LEMMA 8.

$$AP\text{-}\int_0^{2\pi} \{c_1 f_1(x) + c_2 f_2(x)\}\,dx = c_1\left[AP\text{-}\int_0^{2\pi} f_1(x)\,dx\right] + c_2\left[AP\text{-}\int_0^{2\pi} f_2(x)\,dx\right]$$

if the right-hand side exists.

LEMMA 9. *If $f(x)$ is (AP)-integrable over $(0, 2\pi)$ and $g(x) = f(x)$ p.p., then $g(x)$ is (AP)-integrable to the same value, and they have the same second primitive function.*

The proofs of these lemmas follow immediately from the definition (10).

Theorem 6. *Suppose that $f(x)$ is (AP)-integrable on $(0, 2\pi)$ and $F(x)$ is its second primitive. Then $AD^2F(x)$ exists and equals $f(x)$ p.p.*

Proof. Let

$$\pi > \delta > 0, \qquad \epsilon > 0, \qquad k > 0, \qquad \epsilon_1 = \tfrac{1}{16}\delta k\epsilon.$$

Choose upper and lower approximating pairs $\{M_1, \Phi_1(x)\}$, $\{m_1, \phi_1(x)\}$ such that $0 < M_1 - m_1 < \epsilon_1/\pi$. Then $[\Phi_1(x) - F(x)]$ is convex in $(-2\pi, 2\pi)$. By Theorem 2, Corollary,

$$\overline{D}^2[\Phi_1(x) - F(x)] < \infty \quad \text{p.p.};$$

and therefore, by Theorem 3 (ii),

$$A\overline{D}^2[\Phi_1(x) - F(x)] < \infty \quad \text{p.p.}$$

Now

$$\begin{aligned} A\underline{D}^2F(x) &= A\underline{D}^2[\Phi_1(x) - (\Phi_1(x) - F(x))] \\ &\geqslant A\underline{D}^2\Phi_1(x) - A\overline{D}^2[\Phi_1(x) - F(x)] > -\infty \quad \text{p.p.} \end{aligned}$$

Let E_k be the set of points of $(-2\pi+\delta, 2\pi-\delta)$ for which $D^2[\Phi_1(x) - F(x)]$ exists and is at least k. Let F_k be the set for which $A\overline{D}^2[\Phi_1(x) - F(x)] \geqslant k$. Since $[\Phi_1(x) - F(x)]$ is convex, $D^2[\Phi_1(x) - F(x)]$ exists p.p. in $(-2\pi, 2\pi)$. By Theorem 4, $F_k \supset E_k$ and $|F_k| = |E_k|$. Apply Theorem 2 to obtain

$$\frac{\{\Phi_1(2\pi) - F(2\pi)\} - \{\Phi_1(2\pi-\delta) - F(2\pi-\delta)\}}{\delta} - \\ - \frac{\{\Phi_1(-2\pi+\delta) - F(-2\pi+\delta)\} - \{\Phi_1(-2\pi) - F(-2\pi)\}}{\delta} \geqslant \tfrac{1}{4}k|F_k|,$$

i.e.

$$F(2\pi-\delta) - \Phi_1(2\pi-\delta) + F(-2\pi+\delta) - \Phi_1(-2\pi+\delta) \geqslant \tfrac{1}{4}k\delta|F_k|.$$

A fortiori we have

$$\phi_1(2\pi-\delta) - \Phi_1(2\pi-\delta) + \phi_1(-2\pi+\delta) - \Phi_1(-2\pi+\delta) \geqslant \tfrac{1}{4}k\delta|F_k|.$$

But $[\Phi_1(x) - \phi_1(x)]$ is convex and

$$0 \geqslant \Phi_1(0) - \phi_1(0) > -\epsilon_1.$$

Hence

$$\phi_1(2\pi-\delta) - \Phi_1(2\pi-\delta) < 2\epsilon_1, \qquad \phi_1(-2\pi+\delta) - \Phi_1(-2\pi+\delta) < 2\epsilon_1.$$

Thus

$$|F_k| < \frac{16\epsilon_1}{k\delta} = \epsilon.$$

Since ϵ is arbitrary and independent of k, we have $|F_k| = 0$. Suppose now that x is in $(-2\pi+\delta, 2\pi-\delta)$ and not in F_k; then

$$A\underline{D}^2F(x) \geqslant A\underline{D}^2\Phi_1(x) - A\overline{D}^2[\Phi_1(x) - F(x)] \geqslant f(x) - k.$$

Now let k take the values n^{-1} ($n = 1, 2, \ldots$) and put

$$E = \bigcup_{r=1}^{\infty} F_{1/r}.$$

Then $|E| = 0$. If x is not in E and is in $(-2\pi+\delta, 2\pi-\delta)$,

$$A\underline{D}^2F(x) \geqslant f(x), \qquad A\underline{D}^2F(x) > -\infty.$$

Let $\delta \to 0+$, and these inequalities must hold p.p. in $(-2\pi, 2\pi)$. Similarly we have p.p. in $(-2\pi, 2\pi)$

$$A\overline{D}^2F(x) \leqslant f(x) \quad \text{and} \quad A\overline{D}^2F(x) < +\infty.$$

This completes the proof of the theorem.

COROLLARY. *If $f(x)$ is (AP)-integrable, it must be almost everywhere finite-valued.*

A partial converse of Theorem 6 is also true.

THEOREM 7. *Suppose $F(x)$ a function defined in $(-2\pi, 2\pi)$ with the following properties*:

(*a*) *there exists a constant I such that*

$$F(x) - \frac{I}{4\pi}x^2 = \Psi(x)$$

is periodic,

(*b*) *$\Psi(x)$ is Lebesgue-integrable and $\mathscr{A}$-continuous*,

(*c*) *$F(x)$ is approximately continuous and has the property R*,

(*d*) *$AD^2F(x)$ exists and equals $f(x)$ p.p.*,

(*e*) $$-\infty < A\underline{D}^2F(x) \leqslant A\overline{D}^2F(x) < +\infty$$

except in an enumerable set E,

(*f*) *at points of E (if any)*,

$$\lim_{r\to 1-0}\left[(1-r)\frac{\partial^2}{\partial x^2}\Psi(r,x)\right] = 0.$$

Then $f(x)$ is (AP)-integrable over $(0, 2\pi)$, $F(x)$ is its second primitive and

$$I = AP\text{-}\int_0^{2\pi} f(x)\,dx.$$

Proof. The pair $\{I, F(x)\}$ satisfies all the conditions for either an upper or a lower approximating pair.

Remark. Theorem 7 is a partial generalization of Theorem 4.2 of (**3**) since there it is assumed that $F(x)$ is continuous and smooth, and AD^2 is replaced by D^2. The only new condition we have imposed is (*a*), which arises because in our definition of the (AP)-integral we extend the interval of definition of $f(x)$ to make it periodic.

5. Definition of the integral over (a, b)

Suppose that $f(x)$ is defined p.p. in (a, b). Then

$$g(x) = \frac{b-a}{2\pi} f\left[\frac{b-a}{2\pi} x + a\right] \tag{11}$$

is defined p.p. in $(0, 2\pi)$. We say that $f(x)$ is (AP)-*integrable* over (a, b) if and only if $g(x)$ is (AP)-integrable over $(0, 2\pi)$. In this case we write

$$AP\text{-}\int_a^b f(x)\,dx = AP\text{-}\int_0^{2\pi} g(x)\,dx,$$

where $g(x)$ is given by (11).

If $g(x)$ is (AP)-integrable on $(0, 2\pi)$ and $G(x)$ is the second primitive defined by Lemma 7 over $(-2\pi, 2\pi)$. Then

$$F(x) = \frac{b-a}{2\pi} G\left[\frac{2\pi}{b-a}(x-a)\right] \tag{12}$$

is defined for x in $(2a-b, b)$. It is called the *second primitive* of $f(x)$ on this range.

Clearly it would be possible to extend the definition (8) of the generalized Abel second derivates to functions $\Lambda(x)$ which are integrable and of period $\delta > 0$. Then, as in (9), the definition can be extended to functions such that $\Phi(x) - Kx^2$ is periodic and integrable with period δ. If this is done, then Lemmas 8, 9 and Theorems 6, 7 are all true with $(0, 2\pi)$ replaced by a general interval (a, b). The details are left to the reader.

6. The generality of the (AP)-integral

I am going to show that the (AP)-integral includes the special Denjoy integral, which is known to be equivalent to the ordinary Perron integral. I shall again state the results for the interval $(0, 2\pi)$: the corresponding results for a general interval (a, b) are true and can be obtained by the change of scale given by (11). I shall say that $f(x)$ defined p.p. on $(0, 2\pi)$ is *Perron-integrable* there if, given $\epsilon > 0$, there exist continuous functions $M(x)$, $m(x)$ satisfying

$$(a)\ \ M(0) = m(0) = 0, \qquad M(2\pi) - m(2\pi) < \epsilon;$$
$$(b)\ \ \underline{D}M(x) \geqslant f(x) \geqslant \overline{D}m(x)$$

for all x, where $\overline{D}$ is the larger of the two upper Dini derivates, and $\underline{D}$ the smaller of the two lower Dini derivates.

This is not the usual definition of the Perron integral, but it can be seen to be equivalent to it [see (**7**)].

THEOREM 8. *If $f(x)$ is defined p.p. on $(0, 2\pi)$ and is Perron-integrable or Denjoy-special-integrable over that interval, then it is (AP)-integrable to the same value.*

Let
$$I = P\text{-}\int_0^{2\pi} f(x)\, dx.$$

Define $f(x)$ by periodicity on $(-2\pi, 0)$. Let $M(x)$ be a Perron major function defined on $(-2\pi, 0)$ with

$$M(-2\pi) = 0, \qquad \underline{D}M(x) \geqslant f(x)$$

where it is defined, $\qquad \underline{D}M(x) > -\infty$

for all x in $(-2\pi, 0)$, and $M(0) - I < \tfrac{1}{2}\epsilon$. For $0 \leqslant x \leqslant 2\pi$, define

$$M(x) = M(0) + M(x - 2\pi).$$

Then $M(x)$ is a Perron major function for $f(x)$ over $(-2\pi, 2\pi)$, and $M(2\pi) - I < \epsilon$. Let

$$\Phi(x) = \int_{-2\pi}^{x} M(t)\, dt - \frac{x+2\pi}{4\pi} \int_{-2\pi}^{2\pi} M(t)\, dt.$$

Then, if $-2\pi \leqslant x \leqslant 0$,

$$\Phi(x+2\pi) - \Phi(x) = \int_{x}^{x+2\pi} M(t)\, dt - \tfrac{1}{2} \int_{-2\pi}^{2\pi} M(t)\, dt$$

$$= \tfrac{1}{2}\left\{ \int_0^{x+2\pi} M(t)\, dt - \int_{-2\pi}^{x} M(t)\, dt \right\} + \tfrac{1}{2}\left\{ \int_x^0 M(t)\, dt - \int_{x+2\pi}^{2\pi} M(t)\, dt \right\}$$

$$= (x+\pi)M(0) = [(x+2\pi)^2 - x^2]\frac{M(0)}{4\pi}.$$

Thus
$$\Lambda(x) = \Phi(x) - \tfrac{1}{4}\pi^{-1}Mx^2$$

is periodic with $M = M(0)$. It is continuous and differentiable for all x with

$$\Lambda'(x) = M(x) - K - \tfrac{1}{2}\pi^{-1}Mx.$$

Now $M(x) - \tfrac{1}{2}\pi^{-1}Mx$ is periodic and continuous. Let its Fourier series be

$$T + \sum_{n=1}^{\infty} nB_n(x).$$

Then this series is Abel-summable to $M(x) - \tfrac{1}{2}\pi^{-1}Mx$ for all x. Further, since $\underline{D}M(x) \geqslant f(x)$ where it is defined, we have

$$\liminf_{r \to 1-0} \left[-\sum n^2 A_n(x) r^n\right] \geqslant f(x) - \frac{M}{2\pi}.$$

Now a Fourier series can be integrated term by term, so that the series for $\Lambda(x)$ is

$$Tx+\sum_{n=1}^{\infty} A_n(x).$$

Since $\Lambda(x)$ is periodic, we have $T = 0$. Further

$$A\underline{D}^2\Lambda(x)+\frac{M}{2\pi} \geqslant f(x),$$

and hence $$A\underline{D}^2\Phi(x) \geqslant f(x) \text{ p.p.},$$

and $$A\underline{D}^2\Phi(x) > -\infty$$

for all x.

Thus the pair $\{M, \Phi(x)\}$ is an AP upper approximating pair for $f(x)$. Similarly a lower pair $\{m, \phi(x)\}$ can be found. Since $0 < M-m < \epsilon$, which is arbitrary, the theorem is proved.

It is not at present clear whether or not the (AP)-integral includes the (SCP)-integral and the (P^2)-integral. However, I shall show in the next section that the (AP)-integral is more powerful when applied to the 'sum' of trigonometric series. I now show that the (AP)-integral is a 'non-absolute' integral in the same sense as the Perron and Denjoy integrals: that is, it is a real generalization of the Lebesgue integral only in the case where the absolute value of the integrand is not L-integrable.

THEOREM 9. *If $f(x) \geqslant 0$ almost everywhere, and is (AP)-integrable over $(0, 2\pi)$; then $f(x)$ is Lebesgue-integrable to the same value.*

Proof. Let $\{M, \Phi(x)\}$ be an upper approximating pair such that

$$A\underline{D}^2\Phi(x) \geqslant f(x) \geqslant 0 \text{ p.p.}, \qquad A\underline{D}^2\Phi(x) > -\infty$$

for all x. Then, by Theorem 3, we have

$$\bar{D}^2\Phi(x) \geqslant 0 \text{ p.p.}, \qquad \bar{D}^2\Phi(x) > -\infty \quad \text{for all } x.$$

By Theorem 1, $\Phi(x)$ is continuous and convex in $(-2\pi, 2\pi)$. Hence $D^2\Phi(x)$ exists p.p. and is Lebesgue-integrable. But

$$D^2\Phi(x) \geqslant f(x) \geqslant 0 \text{ p.p.},$$

and therefore $f(x)$ is Lebesgue-integrable, and *a fortiori* Perron-integrable. Now apply Theorem 8 to obtain

$$AP\text{-}\int_0^{2\pi} f(x)\,dx = P\text{-}\int_0^{2\pi} f(x)\,dx = L\text{-}\int_0^{2\pi} f(x)\,dx.$$

7. Application to the theory of trigonometric series

For this purpose we need only the (AP)-integral over $(0, 2\pi)$ which we have developed in detail.

Theorem 10. *Suppose that*

$$f(r,x) = a_0 + \sum_{n=1}^{\infty} A_n(x) r^n$$

and
$$a_n = o(n), \qquad b_n = o(n).$$

Let
$$\limsup_{r\to 1-0} f(r,x), \qquad \liminf_{r\to 1-0} f(r,x)$$

be finite except at points of an enumerable set E; *and let*

$$f(x) = \lim_{r\to 1-0} f(r,x)$$

exist and be finite p.p. At points of E *let*

$$\lim_{r\to 1-0} [(1-r)f(r,x)] = 0.$$

Then $f(x)$ *is* (AP)*-integrable over* $(0, 2\pi)$ *and*

$$a_0 = \frac{1}{\pi}\left[AP\text{-}\int_0^{2\pi} f(x)\,dx\right].$$

Proof. Consider the series

$$-\sum_1^{\infty} \frac{A_n(x)}{n^2}.$$

This is a trigonometric series whose coefficients are $o(n^{-1})$, and so, by the Riesz–Fischer theorem, it is the Lebesgue–Fourier series of a function $F(x)$. By Lemma 21 of (**9**), this trigonometric series is Abel-summable for all x to a function with the property R. Hence we may assume that

$$F(x) = \lim_{r\to 1-0}\left[-\sum_1^{\infty} \frac{A_n(x)}{n^2} r^n\right]$$

for all x. Since the coefficients are $o(n^{-1})$, the series is convergent in the ordinary sense to $F(x)$. Hence, by Lemma 9 of (**10**), $F(x)$ is approximately continuous. Now put

$$\Phi(x) = \tfrac{1}{4}a_0 x^2 + F(x) + T,$$

where T is a finite constant chosen so that

$$\Phi(-2\pi) = 0 = \Phi(2\pi).$$

Then the pair $\{\pi a_0, \Phi(x)\}$ forms both an upper and a lower approximating pair for $f(x)$ on $(0, 2\pi)$. Hence $f(x)$ is (AP)-integrable and

$$\pi a_0 = AP\text{-}\int_0^{2\pi} f(x)\,dx.$$

Remark. The conditions imposed in the above theorem are sufficient to imply that the coefficients are uniquely defined. The condition

$$\lim\left[\frac{1}{n^2}\sum_{p=1}^{n} c_p e^{ipx}\right] = 0, \qquad c_n = a_n + ib_n$$

for all x in place of $a_n = o(n)$, $b_n = o(n)$ is also known to imply uniqueness [see (**10**)]. With this less restrictive condition Theorem 10 cannot be proved for the (AP)-integral, as we have defined it in this paper, because the function $F(x)$ is not known to be Lebesgue-integrable. However we might have assumed that the approximating functions were (SCP)-integrable and have built up in the same way an (AP^*)-integral more general than the (AP)-integral. It can be shown that Theorem 10 is true with this integral and the more general uniqueness condition for hypothesis. For the sake of simplicity I did not develop the integral in this way.

I now wish to prove that an Abel-summable trigonometric series is the (AP)-Fourier series of its sum. To do this I shall use the following result on formal multiplication of series.

LEMMA 10. *Suppose* $A_n(x)$, $B_n(x)$ *are defined by* (1), (2),

$$f(r,x) = \sum_{n=1}^{\infty} A_n(x) r^n, \qquad g(r,x) = \sum_{n=1}^{\infty} B_n(x) r^n \quad (0 \leqslant r < 1),$$

and
$$\lim_{r\to 1-0}[(1-r)g(r,x)] = 0$$

for all x. *Let* k *be a positive integer and*

$$\alpha_0 + \sum_{n=1}^{\infty} \mathscr{A}_n(x)$$

be the series obtained by formal multiplication of $\sum A_n(x)$ *by* $\cos kx$;

$$q(r,x) = \alpha_0 + \sum_{n=1}^{\infty} \mathscr{A}_n(x) r^n.$$

Then, at points where $\cos kx \geqslant 0$, *we have*

$$\limsup_{r\to 1-0} q(r,x) = \cos kx \limsup_{r\to 1-0} f(r,x),$$

$$\liminf_{r\to 1-0} q(r,x) = \cos kx \liminf_{r\to 1-0} f(r,x),$$

while, if $\cos kx \leqslant 0$, *then*

$$\limsup_{r\to 1-0} q(r,x) = \cos kx \liminf_{r\to 1-0} f(r,x),$$

$$\liminf_{r\to 1-0} q(r,x) = \cos kx \limsup_{r\to 1-0} f(r,x).$$

This is proved in (12).

THEOREM 11. *If in addition to the hypothesis of Theorem* 10 *we have*

$$\lim_{r\to 1-0}\Big[(1-r)\sum_{n=1}^{\infty} B_n(x)r^n\Big] = 0 \quad \textit{for all } x;$$

then $\qquad f(x)\sin kx, \qquad f(x)\cos kx \quad (k = 1, 2,...)$

are (AP)*-integrable over* $(0, 2\pi)$, *and the given series is the* (AP)*-Fourier series of* $f(x)$

$$a_k = \frac{1}{\pi}\Big[AP\text{-}\int_0^{2\pi} f(x)\cos kx\,dx\Big], \qquad b_k = \frac{1}{\pi}\Big[AP\text{-}\int_0^{2\pi} f(x)\sin kx\,dx\Big].$$

Proof. Multiply the given series formally by $\cos kx$ to give the trigonometric series

$$\alpha_0 + \sum_{n=1}^{\infty} \mathscr{A}_n(x).$$

Then the constant term $\alpha_0 = \frac{1}{2}a_k$. Further at points where

$$f(x) = \lim_{r\to 1-0} f(r, x)$$

exists and is finite, we have, by Lemma 10,

$$f(x)\cos kx = \lim_{r\to 1-0}\Big[\alpha_0 + \sum_1^{\infty} \mathscr{A}_n(x)r^n\Big].$$

At points where $\qquad \limsup_{r\to 1-0},\ \liminf_{r\to 1-0} f(r, x)$

are finite, the same is true of the limits of $\alpha_0 + \sum \mathscr{A}_n(x)r^n$. Finally, it is easy to prove that

$$\lim_{r\to 1-0}[(1-r)(\alpha_0 + \sum \mathscr{A}_n(x)r^n)] = 0$$

for all x. Thus we may apply Theorem 10 to the series $\alpha_0 + \sum_{n=1}^{\infty} \mathscr{A}_n(x)r^n$, and obtain the required result.

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SOME REMARKS ON A MAXIMAL THEOREM OF HARDY AND LITTLEWOOD

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1. LET $f(x)$ be non-negative and integrable in the interval $a \leqslant x \leqslant b$, and let

$$\Theta(x) = \Theta(x,f) = \sup_{a \leqslant \xi < x} \left\{ \frac{1}{x-\xi} \int_{\xi}^{x} f(t)\,dt \right\}. \tag{1.1}$$

The following theorem of Hardy and Littlewood (**2**) is well known:

THEOREM 1. *If $p > 1$, then*

$$\int_a^b \Theta^p(x)\,dx \leqslant A(p) \int_a^b f^p(x)\,dx, \tag{1.2}$$

where $A(p) = \left(\dfrac{p}{p-1}\right)^p$.

This and similar inequalities later are to be interpreted as meaning 'if the integral on the right-hand side is finite, then that on the left is finite and satisfies the inequality'. The value of $A(p)$ here cannot be improved, as is easily seen by taking

$$a = 0, \qquad f = x^{-(1-\epsilon)/p},$$

and making $\epsilon \to 0$.†

Alternative proofs of Theorem 1 have been given by Riesz (**6**) and Rado (**5**), but these follow the original proof of Hardy and Littlewood in so far as (1.2) is deduced by means of Hardy's inequality from a theorem concerning the non-increasing rearrangement of f. Wiener (**7**) has given a more direct proof of (1.2) which does not use either Hardy's inequality or the rearrangement of f, but Wiener's method does not give the best possible value of $A(p)$.‡ By combining the arguments of Riesz and Wiener, however, we can obtain a direct proof of (1.2) which does give the best possible value of $A(p)$, and this is given in § 2 of this note.

† Since f is decreasing, the supremum in (1.1) is here attained when $\xi = 0$.
‡ Wiener's value of $A(p)$ actually tends to infinity with p.

The inequality (1.2) is false when $p = 1$. Hardy and Littlewood (2) showed that in this case

$$\int_a^b \Theta(x)\,dx \leqslant B_1 \int_a^b f(x)\log^+ f(x)\,dx + B_2, \tag{1.3}$$

where B_1 and B_2 depend only on a and b. A direct proof of (1.3) is again given by Wiener (7), but neither Wiener nor Hardy and Littlewood obtain the best possible values of B_1 and B_2. By a refinement of Wiener's argument, I prove here

THEOREM 2. *For any k such that $0 < k < 1$,*

$$\int_a^b \Theta(x)\,dx \leqslant \frac{1}{1-k}\int_a^b f(x)\log^+ f(x)\,dx + \frac{1}{k}(b-a).$$

It should be noted that the coefficient B_1 in (1.3) cannot be less than 1, for, if

$$f(x) = x^{-1}\left(\log\frac{1}{x}\right)^{-2-\epsilon},$$

where $\epsilon > 0$, and if $a = 0$ and b is chosen so small that f is decreasing when $0 < x \leqslant b$, then, as $b \to 0$,

$$\frac{1}{1+\epsilon}\int_a^b f\log^+ f\,dx \sim \int_a^b \Theta\,dx = \frac{1}{\epsilon(1+\epsilon)}\left(\log\frac{1}{b}\right)^{-\epsilon}.$$

In § 3 we consider briefly the relation of these results to Hardy's inequality. In § 4 we discuss the analogues of these results in two dimensions.† The key result in Wiener's discussion of these inequalities is the following

LEMMA.‡ *Let $f(x)$ be non-negative and integrable in the interval $a \leqslant x \leqslant b$, and let*

$$f^*(x) = \sup\left\{\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} f(t)\,dt\right\},$$

where the supremum is taken over all intervals (ξ_1, ξ_2) contained in (a, b) and containing the point x. If E_α is the set of points x of the interval $a < x < b$ for which $f^*(x) > \alpha$, then

$$|E_\alpha| \leqslant \frac{4}{\alpha}\int_{f \geqslant \frac{1}{2}\alpha} f(x)\,dx.$$

† The case of two dimensions is typical, and I shall restrict the discussion to this case.

‡ Wiener deduces the lemma from a result of Sierpinski's. I shall obtain it here from F. Riesz's fundamental lemma.

Now let S denote the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ and suppose that $f(x,y) = f(P)$ is non-negative and integrable in S. For each point P_0 of S let

$$f^*(P_0) = \sup \frac{1}{|I|} \int_I f(P)\, dP, \tag{1.4}$$

the supremum being taken over all intervals I, with sides parallel to the axes, contained in S, and containing P_0. Finally, let E_α denote the set of points P of S for which $f^*(P) > \alpha$.

The immediate analogue of the result of the above lemma, i.e. that

$$|E_\alpha| \leqslant \frac{B}{\alpha} \int_{f \geqslant \frac{1}{2}\alpha} f(P)\, dP, \tag{1.5}$$

where B is some constant, is false. The inequality (1.5) is true, however, if we alter the definition of f^* by requiring that the supremum of the expression on the right of (1.4) be taken only over intervals I for which

$$|I| > \gamma (\text{diameter of } I)^2,$$

where γ is a positive constant (the same for all P_0).† The constant B in (1.5) then depends on γ and tends to infinity as $\gamma \to 0$. For unrestricted choice of the intervals I, Burkill (**1**) has obtained a result corresponding to (1.5) and has used it to give an alternative proof of the theorem of Jessen, Marcinkiewicz, and Zygmund, on the strong differentiability of multiple integrals. Burkill's proof is based on considerations of density, though he comments that his result can also be deduced from the work of Hardy and Littlewood. Wiener's method, however, gives here a simpler proof, and a slightly more complete result. We have in fact

THEOREM 3. *Suppose that $f(x,y) = f(P)$ is non-negative and measurable in the square S, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and that*

$$\int_S f(P) \log^+ f(P)\, dP < \infty.$$

If, for each P_0 of S, $\quad f^*(P_0) = \sup \dfrac{1}{|I|} \displaystyle\int_I f(P)\, dP,$

the supremum being taken over all intervals I, with sides parallel to the axes, contained in S, and containing P_0, and, if E_α is the set of points P of S for which $f^*(P) > \alpha$, then

$$|E_\alpha| \leqslant \frac{16}{\alpha} \int_{f \geqslant \frac{1}{2}\alpha} f(P) \left\{ 1 + \log\left(\frac{4f(P)}{\alpha} \right) \right\} dP.$$

† This result follows trivially from a covering theorem proved by Wiener in (**7**).

The results for two dimensions analogous to Theorems 1 and 2 can be deduced immediately from Theorem 3.

2. We consider first the proofs of Theorems 1 and 2. The kernel of the proofs is contained in the following lemma.

LEMMA 1. *Let $f(x)$ be non-negative and integrable in the interval $a \leqslant x \leqslant b$, and let*

$$\Theta(x) = \sup_{a \leqslant \xi < x} \left\{ \frac{1}{x-\xi} \int_{\xi}^{x} f(t)\, dt \right\}.$$

If G_α is the set of points x of the interval $a < x < b$ for which $\Theta(x) > \alpha$, then

$$|G_\alpha| \leqslant \frac{1}{\alpha} \int_{G_\alpha} f(x)\, dx. \tag{2.1}$$

If further k is any number such that $0 < k < 1$, then also

$$|G_\alpha| \leqslant \frac{1}{(1-k)\alpha} \int_{f \geqslant k\alpha} f(x)\, dx. \tag{2.2}$$

The first inequality of the lemma is the first stage in Riesz's proof of Theorem 1, and is an immediate consequence of Riesz's fundamental lemma applied to the function $\int_a^x f\, dt - \alpha x$.† The second inequality (2.2) is due to Wiener (**7**). To obtain it from (2.1),‡ let $f_1(x)$ be equal to $f(x)$ when $f(x) \geqslant k\alpha$, and to 0 otherwise. Then evidently

$$\Theta(x,f) \leqslant \Theta(x,f_1)+k\alpha,$$

so that G_α is contained in the set in which

$$\Theta(x,f_1) > (1-k)\alpha.$$

By (2.1), the measure of this latter set does not exceed

$$\frac{1}{(1-k)\alpha} \int_a^b f_1\, dx = \frac{1}{(1-k)\alpha} \int_{f \geqslant k\alpha} f\, dx,$$

and this is the required result.

To complete the proofs of Theorems 1 and 2, let $\Theta_N(x)$ be equal to $\Theta(x)$ if $\Theta(x) \leqslant N$, and to N otherwise. Then, if $p \geqslant 1$, we have

$$\int_a^b \Theta_N^p\, dx = -\int_0^N \alpha^p\, d|G_\alpha|,$$

where the integral on the right is a Riemann–Stieltjes integral (for Θ_N is

† Riesz (**6**). ‡ The argument is that of Wiener.

bounded, and the approximate Lebesgue sums for the first integral are approximate Riemann–Stieltjes sums for the second integral). Integrating by parts we obtain

$$\int_a^b \Theta_N^p\,dx = -[\alpha^p|G_\alpha|]_0^N + \int_0^N |G_\alpha|\,d(\alpha^p) \leqslant p\int_0^N \alpha^{p-1}|G_\alpha|\,d\alpha. \tag{2.3}$$

Suppose now that $p > 1$. Then substituting for $|G_\alpha|$ from (2.1) we obtain

$$\int_a^b \Theta_N^p\,dx \leqslant p\int_0^N \alpha^{p-2}\,d\alpha \int_{G_\alpha} f(x)\,dx.$$

By Fubini's theorem, the last expression is equal to

$$p\int_\alpha^b f(x)\,dx \int_0^{\Theta_N(x)} \alpha^{p-2}\,d\alpha = \frac{p}{p-1}\int_a^b f(x)\,\Theta_N^{p-1}(x)\,dx,$$

and, by Hölder's inequality, this does not exceed

$$\frac{p}{p-1}\left\{\int_a^b f^p\,dx\right\}^{1/p}\left\{\int_a^b \Theta_N^p\,dx\right\}^{1-1/p}. \tag{2.4}$$

Since Θ_N is bounded, the last factor in (2.4) is finite, and we may therefore divide by it to obtain

$$\int_a^b \Theta_N^p\,dx \leqslant \left(\frac{p}{p-1}\right)^p \int_a^b f^p\,dx.$$

Making $N \to \infty$, we obtain the result of Theorem 1.

If now $p = 1$, (2.3) gives

$$\int_a^b \Theta_N\,dx \leqslant \int_0^N |G_\alpha|\,d\alpha \leqslant \int_0^\infty = \int_{1/k}^\infty + \int_0^{1/k}$$
$$\leqslant \int_{1/k}^\infty |G_\alpha|\,d\alpha + \frac{1}{k}(b-a).$$

By (2.2), we have

$$\int_{1/k}^\infty |G_\alpha|\,d\alpha \leqslant \frac{1}{1-k}\int_{1/k}^\infty \frac{d\alpha}{\alpha}\int_{f\geqslant k\alpha} f(x)\,dx = \frac{1}{1-k}\int_1^\infty \frac{d\lambda}{\lambda}\int_{f\geqslant\lambda} f(x)\,dx$$
$$= \frac{1}{1-k}\int_{f\geqslant 1} f(x)\,dx \int_1^{f(x)} \frac{d\lambda}{\lambda} = \frac{1}{1-k}\int_{f\geqslant 1} f(x)\log f(x)\,dx$$
$$= \frac{1}{1-k}\int_a^b f(x)\log^+ f(x)\,dx,$$

and again the result follows by a passage to the limit.

3. It seems worth while to remark that Hardy's inequality,†

$$\int_a^b \left\{\frac{1}{x-a}\int_a^x f(t)\,dt\right\}^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)\,dx, \tag{3.1}$$

can easily be deduced from Theorem 1. If f is non-increasing, (3.1) is actually identical with (1.2), for in this case

$$\Theta(x,f) = \frac{1}{x-a}\int_a^x f(t)\,dt.$$

If f is not non-increasing, let $\bar{f}$ be the non-increasing rearrangement of f in the interval $a \leqslant x \leqslant b$. Since

$$\int_a^x f\,dt \leqslant \int_a^x \bar{f}\,dt \quad (a \leqslant x \leqslant b),\ddagger \qquad \int_a^b f^p\,dt = \int_a^b \bar{f}^p\,dt,$$

the inequality (3.1) for f follows immediately from that for $\bar{f}$.

Similar remarks apply to the case $p = 1$. We have therefore proved

THEOREM 4. *For any k such that $0 < k < 1$,*

$$\int_a^b \left\{\frac{1}{x-a}\int_a^x f(t)\,dt\right\} dx \leqslant \frac{1}{1-k}\int_a^b f(x)\log^+ f(x)\,dx + \frac{1}{k}(b-a).$$

4. We come finally to the proof of Theorem 3. We require here a lemma, the first part of which has already been stated in § 1.

LEMMA 2. *Let $f(x)$ be non-negative and integrable in the interval $a \leqslant x \leqslant b$, and let*

$$f^*(x) = \sup\left\{\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} f(t)\,dt\right\},$$

where the supremum is taken over all intervals (ξ_1, ξ_2) contained in (a,b) and containing the point x. If E_α is the set of points x of the interval $a < x < b$ for which $f^*(x) > \alpha$, then

$$|E_\alpha| \leqslant \frac{4}{\alpha}\int_{f \geqslant \frac{1}{2}\alpha} f(x)\,dx.$$

Further,
$$\int_{f^* \geqslant \frac{1}{2}\alpha} f^*(x)\,dx \leqslant 4 \int_{f \geqslant \frac{1}{2}\alpha} f(x)\left\{1+\log\left(\frac{4f(x)}{\alpha}\right)\right\} dx.$$

The first inequality of the lemma is an immediate consequence of the

† See Hardy, Littlewood, and Pólya (**3**), Theorem 327.

‡ Ibid. 277, equation (10.12.2).

second inequality of Lemma 1 with $k = \frac{1}{2}$. To prove the second, we have, as in § 2,

$$\int_{*\geqslant \frac{1}{2}\alpha} f^* \, dx = -\int_{\frac{1}{2}\alpha}^{\infty} \lambda \, d|E_\lambda| \leqslant \int_{\frac{1}{2}\alpha}^{\infty} |E_\lambda| \, d\lambda + \tfrac{1}{2}\alpha |E_{\frac{1}{2}\alpha}|$$

$$\leqslant 4 \int_{\frac{1}{2}\alpha}^{\infty} \frac{d\lambda}{\lambda} \int_{f \geqslant \frac{1}{4}\lambda} f(x) \, dx + 4 \int_{f \geqslant \frac{1}{4}\alpha} f(x) \, dx$$

$$= 4 \int_{f \geqslant \frac{1}{4}\alpha} f(x) \log\left\{\frac{4f(x)}{\alpha}\right\} dx + 4 \int_{f \geqslant \frac{1}{4}\alpha} f(x) \, dx.$$

Suppose now that $f(x, y) = f(P)$ satisfies the hypotheses of Theorem 3, and that, for each P_0 of the unit square S,

$$f^*(P_0) = \sup \frac{1}{|I|} \int_I f(P) \, dP,$$

the supremum being taken over all intervals I, with sides parallel to the axes, contained in S, and containing P_0. Let

$$g(x, y) = \sup\left\{\frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} f(x, v) \, dv\right\},$$

where the supremum is taken over all intervals (η_1, η_2) contained in $(0, 1)$ and containing y, and let

$$h(x, y) = \sup\left\{\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} g(u, y) \, du\right\},$$

the supremum being taken over all intervals (ξ_1, ξ_2) contained in $(0, 1)$ and containing x. Finally, let H_α be the set of points P of S for which $h(P) > \alpha$.

By Theorem 2, with $k = \frac{1}{2}$, we have

$$\iint_S g \, dxdy = \int_0^1 dx \int_0^1 g \, dy \leqslant 2 \int_0^1 dx \int_0^1 f \log^+ f \, dy + 2 < \infty.$$

Thus g is integrable over S, and so is integrable over $0 \leqslant x \leqslant 1$ for almost all y in $0 \leqslant y \leqslant 1$. It follows now from the first part of Lemma 2 that h is finite p.p. in S.

If now (ξ_1, ξ_2) and (η_1, η_2) are linear intervals containing x and y respectively,

$$\frac{1}{(\xi_2 - \xi_1)(\eta_2 - \eta_1)} \int_{\xi_1}^{\xi_2} \int_{\eta_1}^{\eta_2} f(u, v) \, dudv \leqslant \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} g(u, y) \, du \leqslant h(x, y).$$

Hence $f^*(x,y) \leqslant h(x,y)$, so that the set in which $f^* > \alpha$ is contained in H_α.† For a fixed y, it follows from Lemma 2 that the set of points x for which $h(x,y) > \alpha$ has linear measure not exceeding

$$\frac{4}{\alpha} \int\limits_{g \geqslant \frac{1}{2}\alpha} g(x,y)\, dx,$$

whence
$$|H_\alpha| \leqslant \frac{4}{\alpha} \iint\limits_{g \geqslant \frac{1}{2}\alpha} g\, dxdy = \frac{4}{\alpha} \int\limits_0^1 dx \int\limits_{g \geqslant \frac{1}{2}\alpha} g(x,y)\, dy. \tag{4.1}$$

Applying the second inequality of Lemma 2 to the inner integral on the right of (4.1), and integrating from 0 to 1, we obtain the result of Theorem 3.

[*Added in proof*, 26 *November* 1955.] Since the above was written I have been informed by Professor Zygmund that practically the same proof of Theorem 1 was given a few months ago by one of his research students, Mr. Stein, in an unpublished Ph.D. thesis submitted to the University of Chicago.

† Down to this point, the argument is that of Jessen, Marcinkiewicz, and Zygmund (**4**).

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NOTE ON FUNCTIONS OF EXPONENTIAL TYPE IN A HALF-PLANE

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LET $\phi(z) = \log|f(z)|$, where $f(z)$ is regular for $x \geqslant 0$.† Various theorems (**1**) state that, if $f(z)$ is of order 1 in the half-plane, then its rate of growth is governed by its values at a suitable sequence $\{\zeta_n\}$. The present note establishes a new theorem of this type; the conditions imposed on $f(z)$ are strict, but those imposed on $\{\zeta_n\}$ are much relaxed. The present theorem establishes the theorem of 'bounded moments' (**2**), which provides an interesting proof of Titchmarsh's 'convolution' theorem.

THEOREM A. *If* (*a*) $\phi(z) \leqslant cx$ $(x \geqslant 0)$, (*b*) $\{\zeta_n\}$ *is such that* $\sum \cos^2\phi_n/\rho_n$ *diverges*, (*c*) *for every* z *the number of* ζ_n *such that* $|\zeta_n - z| < t$ *does not exceed* $1+Bt$, *where* B *is a constant independent of* z, (*d*) *for all* n,

$$\phi(\zeta_n) \leqslant a\xi_n \quad (a < c);$$

then $\phi(z) \leqslant ax$ *for all* $x \geqslant 0$.

Some condition such as (*c*) is necessary; for $\phi(z)$ is small near any zero of $f(z)$, and the ζ_n must not be allowed to cluster round those zeros. Condition (*b*) may not be best-possible, but it is essential to the present proof.

Condition (*c*) can be restated as follows: let the values of $|\zeta_n - z|$, arranged in non-decreasing order, be $\delta_j(z)$; then for every z,

$$\delta_j(z) \geqslant \beta(j-1),$$

where β is a positive constant. The proof is based on Nevanlinna's representation theorem (**3**) which implies the following:

LEMMA 1. *If* $f(z)$ *is regular for* $x \geqslant 0$, *and if condition* (*a*) *is satisfied, then*

$$\phi(z) = bx - I(z) - S(z),$$

where $b \leqslant c$, *and*

$$I(z) = \frac{x}{\pi}\int\limits_{-\infty}^{\infty} \frac{|\phi(it)|\,dt}{x^2+(y-t)^2}, \qquad S(z) = \sum_{\nu} \log\left|\frac{z+\bar{z}_\nu}{z-z_\nu}\right|,$$

where the z_ν *are the zeros of* $f(z)$ *with* $x_\nu > 0$.

† Throughout this note $z = x+iy = re^{i\theta}$, $\zeta = \xi+i\eta = \rho e^{i\phi}$.

Since $I(z)$ and $S(z)$ are non-negative, $\phi(z) \leqslant bx$ for all $x \geqslant 0$.

Lemma 2. *Let $\phi(z)$ satisfy the conditions of Lemma 1. Let $R(z,\zeta)$ denote the relation*

$$\left|\log\frac{z}{\zeta}\right| \leqslant \alpha,$$

where $0 < \alpha < 1$. Let

$$S_\alpha(z) = \sum_{R(z,z_\nu)} \log\left|\frac{z+\bar{z}_\nu}{z-z_\nu}\right|, \qquad S(z) = S_\alpha(z)+S'_\alpha(z).$$

Then $S'_\alpha = o(x)$, as $r \to \infty$, uniformly in the half-plane.

Proof.
$$\log\left|\frac{z+\bar{z}_\nu}{z-z_\nu}\right| = \tfrac{1}{2}\log\left(1+\frac{4xx_\nu}{|z-z_\nu|^2}\right) \leqslant \frac{2xx_\nu}{|z-z_\nu|^2},$$

so that
$$S'_\alpha(z) \leqslant K(\alpha)\sum_{\nu=1}^{\infty}\frac{xx_\nu}{r^2+r_\nu^2}.$$

By Carleman's theorem $\sum x_\nu/(r^2+r_\nu^2)$ converges uniformly for $r \geqslant 0$; hence it is $o(1)$ as $r \to \infty$.

Lemma 3. *Let $\phi(z)$ satisfy the conditions of Lemma 1, and let*

$$I_\alpha(z) = \frac{x}{\pi}\int\limits_{R(z,it)} \frac{|\phi(it)|\,dt}{x^2+(y-t)^2}, \qquad I(z) = I_\alpha(z)+I'_\alpha(z).$$

Then $I'_\alpha(z) = o(x)$, as $r \to \infty$, uniformly in the half-plane.

Proof.
$$I'_\alpha(z) \leqslant K(\alpha)x\int\limits_{-\infty}^{\infty}\frac{|\phi(it)|\,dt}{r^2+t^2}.$$

The integral converges uniformly for $r \geqslant 1$ (as is implied in Lemma 1, in the case $z = r$) and is therefore $o(1)$ as $r \to \infty$.

We now have, by (*d*), for $\zeta = \zeta_n$,

$$b\xi - I_\alpha(\zeta) - I'_\alpha(\zeta) - S_\alpha(\zeta) - S'_\alpha(\zeta) \leqslant a\xi,$$
$$I_\alpha(\zeta)+S_\alpha(\zeta) \geqslant (b-a)\xi - I'_\alpha(\zeta) - S'_\alpha(\zeta)$$
$$\geqslant d\xi \quad (d > 0)$$

provided that $b > a$, for all large enough ζ_n, by Lemmas 2 and 3. Consequently, if both the hypotheses

$$I_\alpha(\zeta_n) > g\xi_n, \qquad S_\alpha(\zeta_n) > g\xi_n \quad (g > 0)$$

imply the convergence of $\sum \cos^2\phi_n/\rho_n$, then hypothesis (*b*) implies that $b \leqslant a$, which is the theorem to be proved.

LEMMA 4. *If* $\{\zeta_n\}$ *satisfies* (*c*) *and* $I_\alpha(\zeta_n) > g\xi_n$ $(g > 0)$, *then* $\sum \cos^2\phi_n/\rho_n$ *converges.*

LEMMA 5. *If* $\{\zeta_n\}$ *satisfies* (*c*) *and* $S_\alpha(\zeta_n) > g\xi_n$ $(g > 0)$, *then* $\sum \cos^2\phi_n/\rho_n$ *converges.*

Proof of Lemma 4. If $g\xi < I_\alpha(\zeta)$, then

$$\pi g\frac{\xi^2}{\rho^3} \leqslant \frac{\xi^2}{\rho^3}\int\limits_{R(\zeta,it)} \frac{|\phi(it)|\,dt}{\xi^2+(\eta-t)^2} \leqslant K(\alpha)\int\limits_{R(\zeta,it)} \frac{|\phi(it)\,dt}{t^3}.$$

Hence
$$\sum_{\rho_n>1}\frac{\cos^2\phi_n}{\rho_n} \leqslant K(\alpha)\int\limits_{-\infty}^{\infty}\frac{|\phi(it)|}{1+|t|^3}\Big(\sum_{R(\zeta_n,it)}1\Big)\,dt;$$

by condition (*c*) the sum is $O(t)$, and the series converges since

$$\int\limits_{-\infty}^{\infty}\frac{|\phi(it)|}{1+t^2}\,dt < \infty.$$

Proof of Lemma 5. Let ζ_n be 'normal' if

$$|z_\nu-\zeta_n| \leqslant \min(x_\nu, 1)$$

for some ν, say ν_n; and let ζ_n be 'abnormal' otherwise. Then, since ρ_n and $r_{\nu_n}\to\infty$,

$$\frac{\cos^2\phi_n}{\rho_n} \leqslant \frac{\xi_n}{\rho_n^2} \leqslant \frac{2x_{\nu_n}}{(r_{\nu_n}-1)^2} = O\Big(\frac{\cos\theta_{\nu_n}}{r_{\nu_n}}\Big).$$

Hence (if the subscript N denotes the sum over all n for which ζ_n is normal),

$$\sum_N\frac{\cos^2\phi_n}{\rho_n} \leqslant K+K\sum_N\frac{\cos\theta_{\nu_n}}{r_{\nu_n}} \leqslant K+K\sum_{\nu=1}^{\infty}\frac{\cos\theta_\nu}{r_\nu}$$

since by (*c*) the number of n such that $|\zeta_n-z_\nu| \leqslant 1$ is bounded, and so the number of n for which $\nu_n = \nu$ is bounded. Lastly let ζ_n be abnormal. Then

$$g\frac{\xi_n^2}{\rho_n^3} \leqslant \frac{\xi_n}{\rho_n^3}S_\alpha(\zeta_n),$$

and (if the subscript A denotes the sum over all n for which ζ_n is abnormal),

$$\sum_A\frac{\xi_n^2}{\rho_n^3} \leqslant K\sum_A\frac{\xi_n}{\rho_n^3}\sum_{R(\zeta_n,z_\nu)}\log\left|\frac{\zeta_n+\bar z_\nu}{\zeta_n-z_\nu}\right| \leqslant K(\alpha)\sum_\nu\frac{1}{r_\nu^3}T_\nu,$$

where
$$T_\nu = \sum_{A,\,R(\zeta_n,z_\nu)}\xi_n\log\left|\frac{\zeta_n+\bar z_\nu}{\zeta_n-z_\nu}\right|.$$

Now, for each ν, let

$$F_\nu(t) = \max_{|\zeta - z_\nu| = t} \xi \log \left|\frac{\zeta+\bar{z}_\nu}{\zeta-z_\nu}\right|;$$

since both factors of $F_\nu(t)$ are greatest when $\zeta = z_\nu + t$,

$$F_\nu(t) = (x_\nu+t)\log\left(\frac{2x_\nu+t}{t}\right),$$

which is a decreasing function of t for $t > 0$. By condition (c) and the abnormality of the ζ_n,

$$T_\nu \leqslant F_\nu(\min(x_\nu, 1)) + \sum_{j=1}^{K(\alpha)r_\nu} F_\nu(\beta j).$$

Now $\quad F_\nu(x_\nu) = x_\nu \log 3, \qquad F_\nu(1) = (x_\nu+1)\log(2x_\nu+1),$

which are both $\quad O(x_\nu + x_\nu^2) = O(x_\nu r_\nu).$

Also

$$\sum_{j=1}^{Kr_\nu} F_\nu(\beta j) \leqslant \int\limits_0^{Kr_\nu} F_\nu(\beta t)\, dt$$

$$= \int\limits_0^{Kr_\nu} (x_\nu+\beta t)\log\left(\frac{2x_\nu+\beta t}{\beta t}\right) dt$$

$$= x_\nu^2 \int\limits_0^{K\sec\theta_\nu} (1+\beta t)\log\left(1+\frac{2}{\beta t}\right) dt$$

$$= x_\nu^2\, O(\sec\theta_\nu) = O(x_\nu r_\nu).$$

Hence $T_\nu = O(x_\nu r_\nu)$, and so

$$\sum_A \cos^2\phi_n/\rho_n \leqslant K(\alpha) \sum_\nu \cos\theta_\nu/r_\nu < \infty.$$

Thus, $\sum_{n=1}^{\infty} \cos^2\phi_n/\rho_n < \infty$, which proves Lemma 5, and the theorem.

The conclusion of Lemma 5 can be strengthened: $\sum \cos^{1+\delta}\phi_n/\rho_n < \infty$ for all $\delta > 0$; but the present argument does not give a corresponding result for Lemma 4.

From Theorem A we can deduce the following theorem:

THEOREM B. *If* (i) $\phi(z) = \log|f(z)| \leqslant kr$, *where* f *is regular for* $x > 0$, *and* $\displaystyle\int\limits_{-\infty}^{\infty} \frac{|\phi(it)|}{1+t^2}\, dt < \infty$;

(ii) $\{\zeta_n\}$ *satisfies the conditions of Theorem A*;

(iii) $\phi(\zeta_n) \leqslant a\xi_n$ *for all* n;

then $\displaystyle\limsup_{r\to\infty} \frac{1}{r}\phi(re^{i\theta}) \leqslant a\cos\theta$ *for* $|\theta| < \tfrac{1}{2}\pi$.

By Nevanlinna's theorem (**3**)

$$\phi(z) = I_+(z) - I_-(z) - S(z) + bx$$
$$= \phi_1(z) + \phi_2(z),$$

where
$$\phi_1(z) = I_+(z) = \frac{x}{\pi} \int\limits_{-\infty}^{\infty} \frac{\log^+|f(it)|\, dt}{x^2+(y-t)^2} \geqslant 0,$$

and $\phi_2(z) \leqslant bx$, so that ϕ_2 satisfies the conditions of Theorem A. Then

$$\phi_2(\zeta_n) \leqslant \phi(\zeta_n) \leqslant a\xi_n,$$

so that, by Theorem A, $\qquad \phi_2(z) \leqslant ax$

for all $x \geqslant 0$. Now (for example by Lemma 4 with $\alpha < \frac{1}{2}\pi - |\theta|$) $\phi_1(z) = o(r)$ for $|\theta| < \frac{1}{2}\pi$, so that

$$\phi(z) \leqslant ax + o(r), \qquad |\theta| < \tfrac{1}{2}\pi,$$

which is Theorem B.

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ON THE STURM–LIOUVILLE EXPANSION

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1. THE classical theorem, 'Haar's equiconvergence theorem' (**2**), on regular Sturm–Liouville systems is as follows:

Let $q(x)$ be of bounded variation in $(0,\pi)$. Let $\psi_n(x)$ $(n = 0, 1, 2,...)$ be the normalized eigenfunctions of the system

$$\psi''+\{\lambda-q(x)\}\psi = 0, \qquad \psi'(0) = h\,\psi(0), \qquad \psi'(\pi) = H\,\psi(\pi). \tag{1}$$

Let
$$J_N(x,y) = \sum_0^N \psi_n(x)\psi_n(y). \tag{2}$$

Let $\psi_n^0(x)$, $J_N^0(x,y)$, be the corresponding functions for the system with $q(x) \equiv 0$, $h = 0$, $H = 0$, i.e. let

$$\psi_0^0(x) = 1/\surd\pi, \qquad \psi_n^0(x) = \surd(2/\pi)\cos nx \quad (n > 0).$$

Then

(i)
$$J_N(x,y)-J_N^0(x,y) \to 0 \tag{3}$$

boundedly for $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$;

(ii) *for every $f(x)$ of $L(0,\pi)$,*

$$\int_0^\pi \{J_N(x,y)-J_N^0(x,y)\}f(y)\,dy \to 0 \tag{4}$$

uniformly for $0 \leqslant x \leqslant \pi$.

The present note establishes the same results without the hypothesis that $q(x)$ is of bounded variation; it is enough to suppose that $q(x)$ is of $L(0,\pi)$ (this is the minimal hypothesis if we are to use Lebesgue integrals); we replace the differential equation for ψ by either of

$$\psi'(x) = \psi'(0)+\int_0^x \{q(y)-\lambda\}\psi(y)\,dy, \tag{5}$$

$$\psi(x) = \psi(0)+x\,\psi'(0)+\int_0^x \{q(y)-\lambda\}\psi(y)(x-y)\,dy. \tag{6}$$

To establish the theorem under these conditions no new ideas are required; we use the method of contour integration, and the notation, mainly, of E. C. Titchmarsh (**4**).

Quart. J. Math. Oxford (2), 6 (1955), 288-92.

2. First we must show that for all λ, a, A, B, there is a solution $y(x)$ of (5) and (6) such that $y(a) = A$, $y'(a) = B$; and that, for each x, $y(x)$ is an integral function of λ. There is no loss of generality in taking $a = 0$, $x > 0$. Let

$$y_0(x) = A+Bx,$$

and, for $n \geqslant 1$,

$$y_n(x) = y_0(x)+\int\limits_0^x \{q(t)-\lambda\}y_{n-1}(t)(x-t)\,dt.$$

Then

$$v_n(x) = \max_{0\leqslant t\leqslant x}|y_n(t)-y_{n-1}(t)| \leqslant \int\limits_0^x |q(t)-\lambda|v_{n-1}(t)(x-t)\,dt;$$

and, if $|A|+|Bx| \leqslant M$, then

$$v_1(x) \leqslant \int\limits_0^x |q(t)-\lambda|M(x-t)\,dt$$

$$\leqslant Mx\,Q(x,\lambda),$$

where

$$Q(x,\lambda) = \int\limits_0^x |q(t)-\lambda|\,dt.$$

Now by induction $\qquad v_n(x) < M(Qx)^n/n!;$

for

$$\int\limits_0^x |q(t)-\lambda|M\frac{Q^{n-1}t^{n-1}}{(n-1)!}(x-t)\,dt < \frac{MQ^n}{(n-1)!}\max_{0\leqslant t\leqslant x}\{t^{n-1}(x-t)\} < \frac{MQ^nx^n}{n!}.$$

Hence $y_n(x)$ converges to a limit $y(x)$, uniformly for $x\,Q(x,\lambda) \leqslant K$, i.e. uniformly, for each x, in any bounded region of λ. It follows that, for each x, $y(x)$ is an integral function of λ; also, by the bounded convergence of $y_{n-1}(t)$,

$$y(x) = A+Bx+\int\limits_0^x \{q(t)-\lambda\}y(t)(x-t)\,dt$$

$$= A+Bx+\int\limits_0^x du\int\limits_0^u \{q(t)-\lambda\}y(t)\,dt$$

(by Fubini's theorem), so that (6), (5), and the boundary conditions $y(a) = A$, $y'(a) = B$ are all satisfied.

Now let $\phi(x,\lambda)$ be the solution of (6) with $\phi(0) = 1$, $\phi'(0) = h$; let $\chi(x,\lambda)$ be that with $\chi(\pi) = 1$, $\chi'(\pi) = H$; and let

$$\omega(\lambda) = \phi(x)\chi'(x)-\chi(x)\phi'(x).$$

Then, as in Titchmarsh (4), § 1.9,

$$J_N(x,y) = \frac{1}{2\pi i}\int\limits_{Q_N} \chi(x,s^2)\phi(y,s^2)\frac{s\,ds}{\omega(s^2)}, \tag{7}$$

where N' is an integer depending on N, and Q_N is the square with vertices at $(N+\frac{1}{2})(\pm 1 \pm i)$.

3. We now require asymptotic formulae for ϕ, χ, ω, when s is large. Titchmarsh's Lemma 1.7 (**4**), and its proof, still hold without alteration; i.e.

$$\phi(x) = \phi(x, s^2) = \cos sx + h\frac{\sin sx}{s} + \frac{1}{s}\int_0^x \sin s(x-y)\phi(y)q(y)\,dy, \tag{8}$$

$$\phi(x) = O\{e^{|tx|}\} \quad (s = \sigma + it),$$

uniformly in $\arg s$ and uniformly for $0 \leqslant x \leqslant \pi$.

From (8),

$$\phi(x) = \cos sx + h\frac{\sin sx}{s} + \frac{1}{2s}\int_0^x \{\sin sx + \sin s(x-2y)\}q(y)\,dy + O(s^{-2}e^{|tx|})$$

$$= \cos sx + \frac{\sin sx}{s}\left\{h + \tfrac{1}{2}\int_0^x q(y)\,dy\right\} + o\,(s^{-1}e^{|tx|}), \tag{9}$$

provided that
$$\int_0^x \sin s(x-2y)q(y)\,dy = o\,\{e^{|tx|}\}. \tag{10}$$

All the o-terms are uniform in x and in $\arg s$. To prove (10), let $\epsilon > 0$ and let $q_1(x)$ be absolutely continuous and such that

$$\int_0^\pi |q(y) - q_1(y)|\,dy < \epsilon.$$

Then
$$\left|\int_0^x \sin s(x-2y)\{q(y)-q_1(y)\}\,dy\right| < \epsilon\, e^{|tx|},$$

and, by integration by parts,

$$\int_0^x \sin s(x-2y)q_1(y)\,dy = O(s^{-1}e^{|tx|}),$$

all uniformly in x and $\arg s$. By this and similar arguments,

$$\phi(x) = \cos sx + h(x)\frac{\sin sx}{s} + o\,(s^{-1}e^{|tx|}),$$

$$\frac{1}{s}\phi'(x) = -\sin sx + h(x)\frac{\cos sx}{s} + o\,(s^{-1}e^{|tx|}),$$

$$\chi(x) = \cos s(x-\pi) + H(x)\frac{\sin s(x-\pi)}{s} + o\,(s^{-1}e^{|t(x-\pi)|}),$$

$$\frac{1}{s}\chi'(x) = -\sin s(x-\pi) + H(x)\frac{\cos s(x-\pi)}{s} + o\,(s^{-1}e^{|t(x-\pi)|}), \tag{11}$$

uniformly, where

$$h(x) = h+\tfrac{1}{2}\int_0^x q(y)\,dy, \qquad H(x) = H+\tfrac{1}{2}\int_\pi^x q(y)\,dy.$$

Hence also, uniformly in $\arg s$,

$$\omega(s^2) = s\sin\pi s+G\cos\pi s+o(e^{\pi|t|}),$$

where
$$G = H(x)-h(x) = H-h-\tfrac{1}{2}\int_0^\pi q(y)\,dy.$$

Since the system is unaltered by the addition of a constant to $q(x)$, we may take $G = 0$, and then

$$\frac{1}{s}\omega(s^2) = \sin\pi s+o(e^{\pi|t|}). \tag{12}$$

That $N' = N$ follows by applying Rouché's theorem to the functions $\omega(s^2)$ and $s\sin\pi s$ and the contour Q_N.

4. Now from (7), (11), and (12), since $|\sin\pi s| > A\,e^{\pi|t|}$ on Q_N,

$$J_N(x,y) = \frac{1}{2\pi i}\int_{Q_N}\frac{\cos sy\cos s(x-\pi)}{\sin s\pi}\,ds+$$

$$+h(y)\frac{1}{2\pi i}\int_{Q_N}\frac{\sin sy\cos s(x-\pi)}{s\sin s\pi}\,ds + H(x)\frac{1}{2\pi i}\int_{Q_N}\frac{\cos sy\sin s(x-\pi)\,ds}{s\sin s\pi}+$$

$$+o(1),$$

uniformly for $0 \leqslant y \leqslant x \leqslant \pi$.

By Cauchy's theorem the three integrals are respectively equal to

$$\frac{1}{\pi}\left\{1+2\sum_1^N \cos nx\cos ny\right\} = J_N^0(x,y),$$

$$\frac{1}{\pi}\left\{y+2\sum_1^N \frac{1}{n}\cos nx\sin ny\right\} = \int_0^y J_N^0(x,t)\,dt,$$

$$\frac{1}{\pi}\left\{x-\pi+2\sum_1^N \frac{1}{n}\sin nx\cos ny\right\} = -\int_x^\pi J_N^0(t,y)\,dt.$$

By direct evaluation the second and third integrals are both $O(1)$, and both

$$O\left\{\int_0^\infty e^{-|t|(x-y)}\frac{dt}{N}\right\} = O\left\{\frac{1}{N(x-y)}\right\},$$

uniformly in $0 \leqslant y \leqslant x \leqslant \pi$. Since $h(y)$ and $H(x)$ are bounded, we have

$$J_N(x,y)-J_N^0(x,y) = o(1)+O\left\{\frac{1}{1+N|x-y|}\right\},$$

uniformly for $0 \leqslant y \leqslant x \leqslant \pi$, and, by the symmetry of J_N, uniformly for $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$; this proves (3).

For any $f(x)$ of $L(0,\pi)$, uniformly for $0 \leqslant x \leqslant \pi$,

$$\int\limits_0^\pi \{J_N(x,y)-J_N^0(x,y)\}f(y)\,dy = o(1)+O\left\{\int\limits_0^\pi \frac{|f(y)|\,dy}{1+N|x-y|}\right\}$$

$$= o(1)+O\left\{\int\limits_{\max(0,x-N^{-\frac{1}{2}})}^{\min(\pi,x+N^{-\frac{1}{2}})} |f(y)|\,dy\right\},$$

which is $o(1)$ uniformly in $(0,\pi)$, by the uniform continuity of $\int\limits_0^x |f(y)|\,dy$.

5. Similar results, of course, hold with boundary conditions $\psi(0) = 0$, or $\psi(\pi) = 0$, or both.

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ON APPROXIMATE FACTORS OF POLYNOMIALS

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LET $A(z) = a_0+a_1 z+...+a_n z^n$ be a polynomial in a complex variable z, with real coefficients. Then, if certain conditions on the coefficients are satisfied, the polynomial

$$a_m+a_{m+1}z+...+a_{m+\nu}z^\nu \tag{0.1}$$

is approximately a factor of $A(z)$, in the sense that there exists an exact factor of $A(z)$, of degree ν, whose coefficients differ little from those of (0.1).

This paper is concerned with obtaining limits for the errors in such approximate factors, expressed solely in terms of the moduli of the coefficients in $A(z)$. The results obtained are similar in form to certain results obtained by Ostrowski,† but the method used is essentially different and leads to a substantial improvement in the limits and in the conditions for their validity. The novelty of the method lies in the application of a Brouwer fixed-point theorem to demonstrate the existence of a factor whose coefficients lie between certain specified limits.

Defining quantities called *deviations*,

$$D_k = \min_{i,j>0}\{|a_k/a_{k-i}|^{1/i},\ |a_k/a_{k+j}|^{1/j}\} \quad \text{or unity,}$$

whichever is the larger, I shall show that, if

(i) $\qquad D_m,\ D_{m+\nu} \geqslant 6+2\sqrt{6} = 10{\cdot}8...,$

or (ii) $\qquad m+\nu = n \quad \text{and} \quad D_m \geqslant 9,$

there is a factor $\qquad c_m+c_{m+1}z+...+c_{m+\nu}z^\nu$

of $A(z)$ such that $\qquad |c_m-a_m| < \frac{1}{3}|a_m|,$

and (i) $|c_{m+\nu}-a_{m+\nu}| < \frac{1}{3}|a_{m+\nu}|$ or (ii) $c_{m+\nu} = a_{m+\nu}$.

Bounds are obtained for all the coefficients in the factor, which suffice to determine it uniquely.

This result is evidently closely connected with the theorem on the separation of the zeros of a polynomial, proved in R.M.G., that, if

† A. Ostrowski, 'Recherches sur la méthode de Graeffe', *Acta Math.* 72 (1940), 99, denoted below by the initials R.M.G.

D_m, $D_{m+\nu} \geqslant 9$, it is possible to determine two numbers, r_1, r_2 ($r_1 < r_2$) such that m zeros of the polynomial have modulus less than r_1, ν have modulus between r_1 and r_2, and the remaining $n-m-\nu$ have modulus greater than r_2. Indeed, the zeros of the factor in question are precisely those zeros of the polynomial which lie in the annulus $r_1 < |z| < r_2$. One would not expect to obtain an approximation to a factor with D_m or $D_{m+\nu}$ less than 9, for, the condition for separation of the zeros not being satisfied, it would not be possible† to associate a particular set of zeros of $A(z)$ with the polynomial (0·1). The limiting values for D_m and $D_{m+\nu}$, above which bounds are obtained in this paper for the error in the approximate factor, achieve this figure of 9 in one case, and do not greatly exceed it in the more general case.‡

1. Notation and preliminaries

It is convenient from the point of view of notation to write the given polynomial in the form

$$A(z) = z^m \sum_{i=-m}^{n} a_i z^i, \tag{1.1}$$

where we wish to determine the accuracy with which the coefficients of

$$a_0+a_1 z+\ldots+a_\nu z^\nu \quad (\nu \leqslant n)$$

approximate to those of a factor of $A(z)$.

The Newtonian majorant of $A(z)$ is defined, following R.M.G., Ch. I, as the uniquely defined polynomial

$$\mathbf{A}(z) = z^m \sum_{i=-m}^{n} \mathbf{a}_i z^i,$$

where $\mathbf{a}_i$ are positive numbers having the properties:

(i) $\mathbf{a}_i \geqslant |a_i| \quad (-m \leqslant i \leqslant n)$,

(ii) $\mathbf{a}_i^2 \geqslant \mathbf{a}_{i-1}\mathbf{a}_{i+1} \quad (-m < i < n)$,

(iii) there are no smaller numbers having these two properties.

The *slope* R_i and *deviation* D_i are defined as

$$R_i = \mathbf{a}_{i-1}/\mathbf{a}_i \quad (-m < i \leqslant n),$$

$$D_i = R_{i+1}/R_i = \mathbf{a}_i^2/(\mathbf{a}_{i-1}\mathbf{a}_{i+1}) \geqslant 1 \quad (-m < i < n).$$

† The condition for separation is necessary, in the sense that there exists a polynomial (0.1) with any prescribed value of $D_m < 9$ such that the mth and $(m+1)$th roots, when arranged in ascending order of moduli, have in fact equal moduli.

‡ The limits 9 and 10·8... may be compared with the limits given in R.M.G., XXXVI, namely 13·4 and 18·7 respectively.

An index i is said to be *principal* if $D_i > 1$, or if $i = -m$ or n, when it follows that $\mathbf{a}_i = |a_i|$.† It will be assumed throughout that the indices 0, ν are principal.

Defining
$$\alpha = R_0, \qquad \beta = 1/R_{\nu+1}$$
we have
$$|a_i| \leqslant \alpha^{-i}|a_0| \quad (i < 0), \tag{1.2}$$
$$|a_j| \leqslant \beta^{j-\nu}|a_\nu| \quad (j > \nu). \tag{1.3}$$
If $m = 0$ or $\nu = n$, it will be convenient to define $\alpha = 0$ or $\beta = 0$ respectively.

The main results of the paper can now be stated.

THEOREM 1. *If there exist values of δ_0, δ_ν satisfying equations* (3.1) *and such that $0 \leqslant (\delta_0, \delta_\nu) < \frac{1}{3}$, there is a unique factor*
$$c_0+c_1 z+...+c_\nu z^\nu$$
of $A(z)$ such that $|c_i-a_i| < \delta_i \mathbf{a}_i$ $(0 \leqslant i \leqslant \nu)$, where $\delta_i \leqslant \delta_0+\delta_\nu$ are defined when $0 < i < \nu$ by equations (3.3, 3.4).

THEOREM 2. *There exists a unique factor*
$$c_0+c_1 z+...+c_\nu z^\nu$$
of $A(z)$ such that $|c_i-a_i| < \delta_i \mathbf{a}_i$ $(0 \leqslant i \leqslant \nu)$

(i) *if $M = \min(D_0, D_\nu) \geqslant 6+2\surd 6$, and $\delta_0 = \delta_\nu = \theta_1$ is the smaller root of the equation*
$$\theta^2(M+1)+\theta(4-M)+1 = 0,$$
while $\delta_i = 2\theta_1$ $(0 < i < \nu)$;

(ii) *if $\nu = n$, $D_0 \geqslant 9$, and δ_i $(0 \leqslant i < \nu) = \theta_2$, the smaller root of the equation*
$$\theta^2 D_0-\theta(D_0-3)+1 = 0,$$
and $\delta_\nu = 0$.

2. The trial-divisor process

The method of proof is to divide $A(z)$ by a trial divisor
$$c_0+c_1 z+...+c_\nu z^\nu, \tag{2.1}$$
obtaining a penultimate remainder‡
$$c_0^*+c_1^* z+...+c_\nu^* z^\nu.$$

† Various elementary properties of the slope and deviation, including the definition of D_i given in the introduction to this paper, are proved in R.M.G. Such properties as are required I shall, therefore, not prove here; they are, moreover, sufficiently self-evident.

‡ The concept of a penultimate remainder has been used by C. R. Aitken as a basis for a method of successive approximation to a factor of a polynomial. See *Proc. Roy. Soc. Edinburgh*, A 63 (1949), 174–91.

The division is carried out from both ends, to give quotient

$$z^m\{(b_{-m}z^{-m}+\ldots+b_{-1}z^{-1})/c_0+(b_{\nu+1}z+\ldots+b_n z^{n-\nu})/c_\nu\}, \tag{2.2}$$

so that we have the identities

$$a_i = b_i+(b_{i-1}c_1+b_{i-2}c_2+\ldots)/c_0 \quad (i<0), \tag{2.3}$$

$$a_i = c_i^*+(b_{-1}c_{i+1}+b_{-2}c_{i+2}+\ldots)/c_0+ \\ +(b_{\nu+1}c_{i-1}+b_{\nu+2}c_{i-2}+\ldots)/c_\nu \quad (0 \leqslant i \leqslant \nu), \tag{2.4}$$

$$a_i = b_i+(b_{i+1}c_{\nu-1}+b_{i+2}c_{\nu-2}+\ldots)/c_\nu \quad (i>\nu). \tag{2.5}$$

It will be shown that, subject to certain conditions, numbers δ_i $(0 \leqslant i \leqslant \nu)$ can be found such that, if

$$|c_i-a_i| \leqslant \delta_i \mathbf{a}_i \tag{2.6}$$

for each value of i in the range $0 \leqslant i \leqslant \nu$, then

$$|c_i^*-a_i| < \delta_i \mathbf{a}_i. \tag{2.7}$$

Now c_i^* are continuous functions of the set of quantities c_i, provided that c_0 and c_ν do not vanish. There is therefore defined a continuous mapping into itself, $c_i \to c_i^*$, of the region (2.6) of the $(\nu+1)$-dimensional euclidean space with c_i as cartesian coordinates. By a fixed-point theorem due to Brouwer,† it follows that there is a set of values of c_i satisfying the inequalities

$$|c_i-a_i| < \delta_i \mathbf{a}_i \tag{2.8}$$

and such that $c_i^* = c_i$ $(0 \leqslant i \leqslant \nu)$. The trial divisor (2.1) will then be a factor of $A(z)$, with quotient

$$z^m\{(b_{-m}z^{-m}+\ldots+b_{-1}z^{-1})/c_0+1+(b_{\nu+1}z+\ldots+b_n z^{n-\nu})/c_\nu\}.$$

3. The definition of the quantities δ_i

It can be proved, in fact, that, if δ_0 and δ_ν are defined by the equations

$$\left.\begin{aligned} \frac{(1+\delta_0)^2}{(D_0-1)(1-\delta_0)\delta_0} &= 1-\frac{1+\delta_\nu}{(1-\delta_\nu)(M_* D_\nu-1)} \\ \frac{(1+\delta_\nu)^2}{(D_\nu-1)(1-\delta_\nu)\delta_\nu} &= 1-\frac{1+\delta_0}{(1-\delta_0)(M_* D_0-1)} \end{aligned}\right\} \tag{3.1}$$

and satisfy the inequalities

$$0 \leqslant (\delta_0, \delta_\nu) < \tfrac{1}{3}, \tag{3.2}$$

where $$M_* = R_\nu/R_0 \geqslant 1,$$

while ϵ_i $(0 < i \leqslant \nu)$ and δ_i $(0 < i < \nu)$ are defined by

$$(1+\epsilon_i)\left\{1-\frac{(1+\delta_0)R_0}{(1-\delta_0)(R_i-R_0)}-\frac{(1+\delta_\nu)R_i}{(1-\delta_\nu)(R_{\nu+1}-R_i)}\right\} = 1 \tag{3.3}$$

† See, for example, S. Lefschetz, *Algebraic Topology* (A.M.S.), 321.

and $$\delta_i = \frac{1+\delta_0}{1-\delta_0}(1+\epsilon_{i+1})\frac{R_0}{R_{i+1}-R_0}+\frac{1+\delta_\nu}{1-\delta_\nu}(1+\epsilon_i)\frac{R_i}{R_{\nu+1}-R_i}, \tag{3.4}$$

then (2.7) is a consequence of (2.6). It can also be deduced from these equations that

$$\delta_i \leqslant \delta_0+\delta_\nu \quad (0 < i < \nu). \tag{3.5}$$

To prove these results, we commence by supposing that δ_i $(0 \leqslant i \leqslant \nu)$ are arbitrarily chosen positive quantities, except that δ_0, δ_ν are required to satisfy (3.2), and that the coefficients of the trial divisor (2.1) satisfy (2.6).

Let $$\mathbf{c}_i = (1+\delta_i)\mathbf{a}_i \quad (0 \leqslant i \leqslant \nu). \tag{3.6}$$

Then it is not hard to prove by means of the inequalities (1.2, 3) and the identities (2.3–5) that,

(i) if $$|a_0|\delta_0 > \frac{1+\delta_0}{1-\delta_0}(\alpha\mathbf{c}_1+\alpha^2\mathbf{c}_2+\ldots+\alpha^\nu\mathbf{c}_\nu), \tag{3.7}$$

then $$|a_i-b_i| \leqslant \delta_0\,\alpha^{-i}|a_0| \quad (-m \leqslant i < 0); \tag{3.8}$$

(ii) if $$|a_\nu|\delta_\nu > \frac{1+\delta_\nu}{1-\delta_\nu}(\beta\mathbf{c}_{\nu-1}+\beta^2\mathbf{c}_{\nu-2}+\ldots+\beta^\nu\mathbf{c}_0), \tag{3.9}$$

then $$|a_i-b_i| \leqslant \delta_\nu\beta^{i-\nu}|a_\nu| \quad (\nu < i \leqslant n); \tag{3.10}$$

(iii) if (3.8) and (3.10) are satisfied and if

$$|\mathbf{a}_i|\delta_i > \frac{1+\delta_0}{1-\delta_0}(\alpha\mathbf{c}_{i+1}+\alpha^2\mathbf{c}_{i+2}+\ldots+\alpha^{\nu-i}\mathbf{c}_\nu)+ \\ +\frac{1+\delta_\nu}{1-\delta_\nu}(\beta\mathbf{c}_{i-1}+\beta^2\mathbf{c}_{i-2}+\ldots+\beta^i\mathbf{c}_0), \tag{3.11}$$

then the inequalities (2.7) are satisfied.

We next simplify the three sets of inequalities (3.7, 3.9, 3.11) by replacing the series in their right-hand members by suitably chosen geometric series. Suppose that ϵ_i $(0 < i \leqslant \nu)$ are positive numbers such that

$$\mathbf{c}_j \leqslant \begin{cases}(1+\epsilon_{i+1})\mathbf{a}_i(R_{i+1})^{i-j} & (0 \leqslant i < j \leqslant \nu),\\ (1+\epsilon_i)\mathbf{a}_i(R_i)^{i-j} & (0 \leqslant j < i \leqslant \nu).\end{cases} \tag{3.12}$$

Then†

$$\alpha\mathbf{c}_{i+1}+\alpha^2\mathbf{c}_{i+2}+\ldots+\alpha^{\nu-i}\mathbf{c}_\nu \leqslant (1+\epsilon_{i+1})\mathbf{a}_i(\alpha R_{i+1}^{-1}+\alpha^2R_{i+1}^{-2}+\ldots) \\ < (1+\epsilon_{i+1})\mathbf{a}_i\,R_0/(R_{i+1}-R_0), \text{ since } \alpha = R_0.$$

and, similarly,

$$\beta\mathbf{c}_{i-1}+\beta^2\mathbf{c}_{i-2}+\ldots+\beta^i\mathbf{c}_0 < (1+\epsilon_i)\mathbf{a}_i\,R_i/(R_{\nu+1}-R_i).$$

† Note that $R_{i+1}/\alpha = R_{i+1}/R_0 > R_1/R_0 > 1$.

If, then, it is possible to define δ_i $(0 \leqslant i \leqslant \nu)$ and ϵ_i $(0 < i \leqslant \nu)$ so that (3.2), (3.4), and (3.12) are satisfied, and so that also

$$\left.\begin{aligned} \delta_0 &= \frac{1+\delta_0}{1-\delta_0}(1+\epsilon_1)\frac{R_0}{R_1-R_0} \\ \text{and} \qquad \delta_\nu &= \frac{1+\delta_\nu}{1-\delta_\nu}(1+\epsilon_\nu)\frac{R_\nu}{R_{\nu+1}-R_\nu} \end{aligned}\right\}, \tag{3.13}$$

the inequalities (3.7, 3.9, 3.11) will follow from (2.6), so that (2.7) will also follow from (2.6) and there will exist a factor (2.1) of the given polynomial whose coefficients satisfy (2.8).

It turns out that the definition of δ_i, ϵ_i is completed by adjoining equation (3.3) to (3.4) and (3.13), when (3.12), (3.5) are consequences of (3.2), but the details of the proof seem to have no particular interest. The form of (3.3) is, however, suggested by considering the case where all the R_i $(0 < i \leqslant \nu)$ are equal, when the required conditions are satisfied with δ_i all equal and $\epsilon_i = \delta_i$; (3.4) then leads immediately to (3.3).

Finally, the equations (3.1) may replace (3.12), being the result of eliminating ϵ_1, ϵ_ν between the latter and (3.3). Theorem 1, apart from the uniqueness of the factor, follows immediately from these results.

4. Proof of Theorem 2

We now require a lemma, whose proof I omit.

LEMMA. *If the equations* (3.1) *have, for given values of the parameters* D_0, D_ν, M_*, *a solution satisfying* (3.2), *and if one or more of these parameters is replaced by a larger value, then they still have such a solution, and one for which both* δ_0 *and* δ_ν *are smaller in value.*

From this lemma, it follows that, if the equations

$$\frac{(1+\delta_0)^2}{(1-\delta_0)\delta_0}+\frac{1+\delta_\nu}{1-\delta_\nu} = M-1 = \frac{(1+\delta_\nu)^2}{(1-\delta_\nu)\delta_\nu}+\frac{1+\delta_0}{1-\delta_0},$$

obtained from (3.1) by replacing D_0, D_ν by $M = \min(D_0, D_\nu)$ and M_* by its least possible value of unity, have a solution satisfying (3.2), then there is a solution of (3.1), satisfying (3.2), with equal or smaller values for both δ_0 and δ_ν. The symmetry of these new equations now suggests that we seek solutions for which $\delta_0 = \delta_\nu = \theta$, leading to

$$\frac{(1+\theta)^2}{(1-\theta)\theta}+\frac{1+\theta}{1-\theta} = M-1,$$

i.e.

$$\theta^2(M+1)+\theta(4-M)+1 = 0.$$

This equation has a positive real root θ_1 not greater than $\frac{1}{3}$ if and only if $M \geqslant 6+2\sqrt{6}$. Together with the inequality (3.5) this establishes Theorem 2 (i), apart from uniqueness.

The proof of Theorem 2 (ii) is similar. Since $\nu = n$, the identity (2.5) does not occur, while certain terms disappear from (2.4). It is found that $\delta_\nu = 0$, and that the equation defining δ_0 is obtained from the general equation (3.1) by omitting terms containing δ_ν: thus

$$\frac{(1+\delta_0)^2}{(D_0-1)(1-\delta_0)\delta_0} = 1.$$

This equation can be written in the form

$$D_0\delta_0^2-(D_0-3)\delta_0+1 = 0,$$

which has a real positive root θ_2 not greater than $\frac{1}{3}$ if and only if $D_0 \geqslant 9$. Also, from (3.5), which remains valid in this special case, $\delta_i \leqslant \delta_0$. This establishes Theorem 2 (ii), again apart from uniqueness.

5. The uniqueness of the factor

Since $D_0, D_\nu \geqslant 9$, it follows immediately from R.M.G. XXI, that $A(z)$ has at most ν zeros in the region $r_1 < |z| < r_2$, where r_1 is the smaller root of

$$2r^2-(3R_0+R_1)r+2R_0 R_1 = 0, \tag{5.1}$$

and r_2 is the larger root of

$$2r^2-(3R_\nu+R_{\nu+1})r+2R_\nu R_{\nu+1} = 0.$$

It can be shown, by means of the inequalities (2.8) satisfied by the coefficients in the factor, that all the zeros of the factor lie in this region. This identifies its zeros, and thereby establishes its uniqueness.

I give the proof in the case of Theorem 1; the proof for the other theorems is similar. Under the conditions of this theorem, it follows from (2.8) and (3.12) that

$$\begin{aligned}|c_0+c_1z+\ldots+c_\nu z^\nu| &> |a_0|\Big\{(1-\delta_0)-(1+\epsilon_1)\sum_{i=1}^{\nu} R_1^{-i}|z|^i\Big\}\\ &> |a_0|\{(1-\delta_0)-(1+\epsilon_1)|z|/(R_1-|z|)\}.\end{aligned}$$

Now, from (3.2, 3.13),

$$(1+\epsilon_1)/(1-\delta_0) = \delta_0(D_0-1)/(1+\delta_0) \leqslant \tfrac{1}{4}(D_0-1).$$

Hence, if z is a zero of the factor,

$$|z|/(R_1-|z|) > 4/(D_0-1),$$

whence

$$|z| > 4R_1/(D_0+3).$$

But the sum of the reciprocals of the roots of (5.1) is $(D_0+3)/2R_1$, so that

$$|z| > 4R_1/(D_0+3) \geqslant r_1.$$

Similarly,

$$|z| < r_2.$$

It may be remarked, finally, that the division process, as well as the definition of the Newtonian majorant and its properties, may be extended to the case where the polynomial $A(z)$ is replaced by a Laurent series, possibly terminating in one direction, and that both theorems remain true, the proof requiring only minor modification.

CONNEXIONS FOR PARALLEL DISTRIBUTIONS IN THE LARGE

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1. IT is well known that on a manifold with a symmetric affine connexion every parallel distribution (field of parallel planes) is integrable.† In the present note I discuss the converse problem and show that for any system of distributions there exists an affine connexion in the large with respect to which the distributions are parallel and which is symmetric if the system is integrable. It is a simple matter to establish the existence of an asymmetric affine connexion, whether the given distributions are integrable or not; there are many such connexions and the main problem is to show that there is among them one that is symmetric when the system of distributions satisfies certain integrability conditions.‡

We shall be concerned only with manifolds and distributions of class C^∞ since we need certain existence theorems which are known to be true for all such manifolds but have not yet been established for analytic manifolds.

2. From properties of distributions discussed in previous papers we know that, if D and D' are any two disjoint distributions on a manifold M, their sum $D+D'$ is also a distribution and is parallel with respect to a given connexion if D and D' are parallel. We shall now say that distributions $D_1,\dots, D_m$ form a *complete system* if they are disjoint and if their sum is the distribution of tangent planes of M.

If D, $\bar{D}$ form a complete system, i.e. are disjoint and complementary, there are associated with them two second-order mixed tensors $\mathbf{a} = (a^i_j)$, $\bar{\mathbf{a}} = (\bar{a}^i_j)$ at every point of M with the properties, in matrix notation,

$$\mathbf{a}^2 = \mathbf{a}, \qquad \bar{\mathbf{a}}^2 = \bar{\mathbf{a}}, \qquad \mathbf{a}\bar{\mathbf{a}} = \bar{\mathbf{a}}\mathbf{a} = \mathbf{0}, \qquad \mathbf{a}+\bar{\mathbf{a}} = \mathbf{I}. \tag{1}$$

† For an account of parallel distributions see A. G. Walker, *Quart. J. of Math.* (Oxford) 20 (1949), 135–45, where results are given for a Riemannian manifold but can be applied also to a manifold with affine connexion. See also Schouten *Ricci-Calculus* (2nd ed., Berlin 1954). The term *involutory* is used by some writers in place of my *integrable*; see for example Chevalley, *Theory of Lie Groups* (Princeton, 1946).

‡ The problem for a single distribution has been solved independently by T. J. Willmore using a different method from ours. (*Proc. London Math. Soc.* in press.)

Also, if $\dim D = r$ and $\dim \bar{D} = \bar{r}$, so that $r+\bar{r} = n = \dim M$, then

$$\operatorname{rank} \mathbf{a} = r, \qquad \operatorname{rank} \bar{\mathbf{a}} = \bar{r}. \tag{2}$$

At any point of M the r-plane belonging to D is the set of vectors $\boldsymbol{\lambda}$ which satisfy the matrix equation $\bar{\mathbf{a}}\boldsymbol{\lambda} = \mathbf{0}$, equivalent to $(\mathbf{a}-\mathbf{I})\boldsymbol{\lambda} = \mathbf{0}$. The $\bar{r}$-plane of $\bar{D}$ is similarly given by $\mathbf{a}\boldsymbol{\lambda} = \mathbf{0}$, so that D and $\bar{D}$ are both determined by the tensor field $\mathbf{a}$.

To see how $\mathbf{a}$ and $\bar{\mathbf{a}}$ are determined at every point of M by the pair $(D, \bar{D})$, let
$$\lambda^i_{\alpha'},\ \lambda^i_{\alpha''} \quad (\alpha' = 1,\dots, r;\ \alpha'' = r+1,\dots, n)$$
be base vectors for the planes of D and $\bar{D}$ respectively at any point x. If Λ is the $n\times n$ matrix (λ^i_α), Λ^{-1} exists since D and $\bar{D}$ are disjoint, and we can write $\Lambda^{-1} = (\mu^\alpha_i)$. The tensors $\mathbf{a}$, $\bar{\mathbf{a}}$ at x are now given by

$$a^i_j = \sum_{\alpha'=1}^{r} \lambda^i_{\alpha'} \mu^{\alpha'}_j, \qquad \bar{a}^i_j = \sum_{\alpha''=r+1}^{n} \lambda^i_{\alpha''} \mu^{\alpha''}_j,$$

and it can easily be verified that they are independent of the particular bases chosen for the two planes at x. Properties (1) and (2) now follow at once.

The representation of distributions by tensor fields can be extended to any complete system $\{D_\sigma\}$ $(\sigma = 1,\dots, m)$, and there is associated with each member D_σ a tensor $\underset{\sigma}{\mathbf{a}} = \big(\underset{\sigma}{a^i_j}\big)$ at every point of M with the properties

(i) $\underset{\sigma}{\mathbf{a}}^2 = \underset{\sigma}{\mathbf{a}}, \qquad \underset{\rho}{\mathbf{a}}\underset{\sigma}{\mathbf{a}} = \mathbf{0} \quad (\rho \neq \sigma), \qquad \sum_{\rho=1}^{m} \underset{\rho}{\mathbf{a}} = \mathbf{I};$ (3)

(ii) $\operatorname{rank} \underset{\sigma}{\mathbf{a}} = \dim D_\sigma$;

(iii) the vectors $\boldsymbol{\lambda}$ in the plane of $\underset{\sigma}{D}$ are given by $\big(\underset{\sigma}{\mathbf{a}}-\mathbf{I}\big)\boldsymbol{\lambda} = \mathbf{0}$.

The tensor $\underset{\sigma}{\mathbf{a}}$ is defined exactly as before, being the first tensor associated with the disjoint complementary pair $(D_\sigma, \bar{D}_\sigma)$, where $\bar{D}_\sigma = \sum_{\rho\neq\sigma} D_\rho$. We write $\underset{\sigma}{\bar{\mathbf{a}}}$ for $\mathbf{I} - \underset{\sigma}{\mathbf{a}}$, i.e. for the second tensor associated with the pair $(D_\sigma, \bar{D}_\sigma)$.

3. Suppose now that an r-dimensional distribution D is given, and let $\bar{D}$ be any $(n-r)$-dimensional distribution such that D, $\bar{D}$ are disjoint and complementary. That such a distribution (of class C^∞) exists over the whole manifold is certainly true; it could, for example, be the orthogonal complement of D with respect to a positive definite metric tensor of class C^∞ since such tensors are known to exist. Writing $\mathbf{a}$, $\bar{\mathbf{a}}$ for the tensors associated with $(D, \bar{D})$, we wish to find the conditions to be satisfied by $\mathbf{a}$ for D to be integrable, and also the conditions for D to be parallel with respect to a given connexion.

With the notation of § 2, let $\lambda^i_{\alpha'}$, $\lambda^i_{\alpha''}$ be differentiable basis vectors for D and $\bar{D}$ defined over a coordinate neighbourhood of M. Then it is well known that D is integrable if

$$\lambda^p_{\beta'}\lambda^q_{\gamma'.p}-\lambda^p_{\gamma'}\lambda^q_{\beta'.p} = \phi^{\alpha'}_{\beta'\gamma'}\lambda^q_{\alpha'}$$

for some ϕ's, where a dot denotes partial differentiation. Multiplying by $\bar{a}^i_q\mu^{\beta'}_j\mu^{\gamma'}_k$ and summing for $q = 1,..., n$ and $\beta', \gamma' = 1,..., r$, it can be verified that the conditions are equivalent to

$$(a^i_{p.q}-a^i_{q.p})a^p_j a^q_k = 0. \tag{4}$$

These, then, are the conditions that $\mathbf{a}$ must satisfy for D to be integrable. It should be noted that they do not imply the corresponding conditions for $\bar{\mathbf{a}}$; the integrability of D does not imply the integrability of $\bar{D}$.

Suppose now that D is parallel with respect to a given connexion $L = (L^i_{jk})$. Then, if covariant differentiation with respect to L is denoted by a vertical line, we have†

$$\lambda^p_{\beta'|k} = A^{\gamma'}_{\beta'k}\lambda^p_{\gamma'}$$

for some A's. Multiplying by $\bar{a}^i_p\mu^{\beta'}_j$, and summing for $p = 1,..., n$ and $\beta' = 1,..., r$, it can be verified that these conditions are equivalent to

$$\bar{a}^i_p a^p_{j|k} = 0. \tag{5}$$

These are the conditions that the associated tensors must satisfy for D to be parallel. Since $\mathbf{a}$ and $\bar{\mathbf{a}}$ cannot in general be interchanged in these equations, we see that, if D is parallel, it does not necessarily follow that $\bar{D}$ is also parallel.

From the identities $\bar{\mathbf{a}}\mathbf{a} = \mathbf{0}$, $\mathbf{a}+\bar{\mathbf{a}} = \mathbf{I}$, equations (5) are seen to be equivalent to

$$a^i_{p|k}a^p_j = 0. \tag{5'}$$

4. Suppose that D is a given distribution and that we wish to find a connexion L with respect to which D is parallel and which is symmetric if D is integrable. We first choose any symmetric connexion $\Gamma = (\Gamma^i_{jk})$ defined over M. Such a connexion always exists in a manifold of class C^∞; it could, for example, be the set of Christoffel symbols given by a metric tensor. Since the difference between two connexions is a tensor, L will be of the form

$$L = \Gamma+T, \tag{6}$$

where $T = (T^i_{jk})$ is a tensor which remains to be determined.

Let $\bar{D}$ be such that D, $\bar{D}$ are disjoint and complementary, and let $\mathbf{a}$, $\bar{\mathbf{a}}$ be the tensors associated with $(D, \bar{D})$. Then, if a comma and a vertical line denote covariant differentiation with respect to Γ and L respectively, we have the identity

$$a^p_{j|k} = a^p_{j,k}+a^q_j T^p_{qk}-a^p_q T^q_{jk}. \tag{6'}$$

† A. G. Walker, loc. cit.

Substituting in (5), we see that D is parallel with respect to L if T satisfies the equations

$$\bar{a}^i_p a^q_j T^p_{qk} = -\bar{a}^i_p a^p_{j,k} = -a^i_{p,k} a^p_j. \tag{7}$$

Here we have used the identities

$$\bar{\mathbf{a}}\mathbf{a} = 0, \qquad \mathbf{a}+\bar{\mathbf{a}} = \mathbf{I}, \qquad \delta^i_{j|k} = 0.$$

Another consequence of these identities is

$$a^i_p a^q_j a^p_{q,k} = 0, \tag{8}$$

and because of this we see that one solution of equations (7) for T is

$$T^i_{jk} = -\bar{a}^i_p a^p_{j,k}.$$

The general solution of (7) is now seen to be

$$T^i_{jk} = -\bar{a}^i_p a^p_{j,k} + V^i_{jk}, \tag{9}$$

where V is any tensor satisfying

$$\bar{a}^i_p a^q_j V^p_{qk} = 0. \tag{10}$$

Substituting in (6) we therefore have the most general connexion with respect to which D is parallel.

If D is integrable, then (4) is satisfied, and, since Γ is symmetric and $\mathbf{a}^2 = \mathbf{a}$, these conditions can be expressed in the form

$$a^i_{p,q} a^p_j a^q_k = a^i_{p,q} a^p_k a^q_j. \tag{11}$$

We want to choose T so that, when these conditions are satisfied, L is symmetric; T will then be symmetric, and hence, from (9), V will satisfy

$$V^i_{jk} - V^i_{kj} = \bar{a}^i_p (a^p_{j,k} - a^p_{k,j}) \tag{12}$$

in addition to (10). Because of (8) and (11) a solution of equations (10) and (12) is at once seen to be

$$V^i_{jk} = -\bar{a}^i_p a^p_{k,j} + a^i_{p,q} a^p_k a^q_j.$$

Substituting in (9) and using the identities $\bar{\mathbf{a}}\mathbf{a} = \mathbf{0}$, $\mathbf{a}+\bar{\mathbf{a}} = \mathbf{I}$, we can write

$$T^i_{jk}(\mathbf{a}, \Gamma) = -a^i_{p,j} a^p_k - a^i_{p,k} a^p_j + a^i_{p,q} a^p_k a^q_j. \tag{13}$$

The connexion $$L(\mathbf{a}, \Gamma) = \Gamma + T(\mathbf{a}, \Gamma)$$

thus satisfies our requirements; it is of class C^∞, it makes D parallel, and it is symmetric when D is integrable.

It should be noted that the connexion $L(\mathbf{a}, \Gamma)$ is defined over the whole manifold. The tensor field $\mathbf{a}$ associated with $(D, \bar{D})$ exists in the large since it is independent of the particular choice of basis vectors used to define it locally, and the expression for L in terms of $\mathbf{a}$ and Γ holds in every allowable coordinate system at every point of M. Thus L depends only upon M, D and whatever distribution $\bar{D}$ and symmetric connexion Γ are chosen over M.

The connexion $L(\mathbf{a},\Gamma)$ derived above is not the only solution of our problem, the most general being $L(\mathbf{a},\Gamma)+W$, where W satisfies

$$W^i_{jk} = W^i_{kj}, \qquad \bar{a}^i_p a^q_j W^p_{qk} = 0.$$

Another form for the general solution is obtained by noting that every solution is given by $L(\mathbf{a},\Gamma)$ for some symmetric connexion Γ. For suppose D integrable and L any symmetric connexion making D parallel. Then taking L in place of the previous Γ, as of course we may do, we see from (13) and (5′) that $T(\mathbf{a}, L) = 0$, so that $L(\mathbf{a}, L) = L$, i.e. L is given by $L(\mathbf{a},\Gamma)$ with a suitable Γ.

If now we regard our chosen Γ as fixed, and let $C = (C^i_{jk})$ be an arbitrary symmetric tensor field over M, the general symmetric connexion making D parallel is obtained by putting $\Gamma+C$ in place of Γ in $L(\mathbf{a},\Gamma)$, i.e. is

$$\Gamma+C+T(\mathbf{a},\Gamma+C).$$

5. The above results can be generalized to any number of distributions. Suppose that disjoint distributions $D_1,\ldots, D_m$ $(m > 1)$ are given over M and that we wish to find a connexion L which makes them parallel. Suppose also that L is required to be symmetric when certain integrability conditions are satisfied. To see what these conditions must be, we observe that, if L is symmetric, then not only are $D_1,\ldots, D_m$ integrable but so also is the sum of any distributions of the system. On the other hand, the integrability of $D_1,\ldots, D_m$ does not imply that a sum of distributions of the system is integrable, for, if D, D' are integrable, $D+D'$ is not necessarily integrable. We shall say that a system $\{D_\sigma\}$ $(\sigma = 1,\ldots, m)$ is integrable if every sum of distributions of the system is integrable. To express the conditions of integrability of a system as simply as possible we write D, D^σ $(\sigma = 1,\ldots, m)$ for the sums

$$D = \sum_{\rho=1}^{m} D_\rho, \qquad D^\sigma = \sum_{\rho\neq\sigma} D_\rho.$$

Then every sum of distributions of the system $\{D_\sigma\}$ appears as an intersection of distributions of the system $\{D, D^\sigma\}$, and, since the intersection of integrable distributions is integrable, we see that the system $\{D_\sigma\}$ is integrable if and only if $D, D^1,\ldots, D^m$ are integrable. Our problem now is to find a connexion L which makes $D_1,\ldots, D_m$ parallel and is symmetric when $D, D^1,\ldots, D^m$ are integrable.

As before we choose a distribution $\bar{D}$ so that D and $\bar{D}$ are disjoint and complementary. Write

$$\bar{D}_\sigma = \bar{D}+D^\sigma, \qquad \bar{D}^\sigma = \bar{D}+D_\sigma.$$

Then D_σ, $\bar{D}_\sigma$ are disjoint and complementary, and so also are D^σ, $\bar{D}^\sigma$.

The system $\{D_\sigma, \bar{D}\}$ is complete and therefore determines tensors $\underset{\sigma}{\mathbf{a}}$, $\bar{\mathbf{a}}$ $(\sigma = 1,..., m)$ over M. We write $\mathbf{a} = \sum_{\rho=1}^{m} \underset{\rho}{\mathbf{a}}$, so that $\mathbf{a}$, $\bar{\mathbf{a}}$ are the tensors associated with the pair $(D, \bar{D})$. We also write

$$\underset{\sigma}{\mathbf{b}} = \sum_{\rho \neq \sigma} \underset{\rho}{\mathbf{a}} = \mathbf{a} - \underset{\sigma}{\mathbf{a}}, \qquad \underset{\sigma}{\bar{\mathbf{b}}} = \bar{\mathbf{a}} + \underset{\sigma}{\mathbf{a}}, \tag{14}$$

so that $\underset{\sigma}{\mathbf{b}}$, $\underset{\sigma}{\bar{\mathbf{b}}}$ are the tensors associated with the pair $(D^\sigma, \bar{D}^\sigma)$.

We again choose a symmetric connexion Γ over M and write L^* for the connexion

$$L^* = L(\mathbf{a}, \Gamma)$$

as defined in § 4, so that L^* is symmetric if D is integrable. A semicolon will denote covariant differentiation with respect to L^*.

Consider the connexion

$$L = L^* + \sum_{\rho=1}^{m} T\left(\underset{\rho}{\mathbf{b}}, L^*\right), \tag{15}$$

where $T\left(\underset{\rho}{\mathbf{b}}, L^*\right)$ is given by (13) with $\underset{\rho}{\mathbf{b}}$ in place of $\mathbf{a}$ and a semicolon in place of every comma. If D is integrable, L^* is symmetric, and, if also $D^1,..., D^m$ are integrable, then $T\left(\underset{\rho}{\mathbf{b}}, L^*\right)$ $(\rho = 1,..., m)$ and hence L are symmetric. Thus L is symmetric if the given system $\{D_\sigma\}$ is integrable. I shall now show that the distributions $D^1,..., D^m$ are parallel with respect to L; and, since each D_σ is an intersection of distributions of the system $\{D^\sigma\}$, it will follow that $D_1,..., D_m$ are parallel with respect to L, i.e. L is a connexion satisfying all our requirements.

From (5), D^σ is parallel with respect to L if

$$\underset{\sigma}{\bar{b}^i_p}\, \underset{\sigma}{b^p_{j|k}} = 0,$$

and on substituting from (15) this becomes, as in the derivation of (7),

$$\underset{\sigma}{\bar{b}^i_p}\, \underset{\sigma}{b^q_j} \sum_{\rho=1}^{m} T^p_{qk}\left(\underset{\rho}{\mathbf{b}}, L^*\right) = -\underset{\sigma}{\bar{b}^i_p}\, \underset{\sigma}{b^p_{j;k}}.$$

When ρ is restricted to the value σ, this equation is satisfied because, from the result of § 4, D^σ is parallel with respect to $L^* + T\left(\underset{\sigma}{\mathbf{b}}, L^*\right)$. It therefore remains for us to prove that

$$\underset{\sigma}{\bar{b}^i_p}\, \underset{\sigma}{b^q_j} \sum_{\rho \neq \sigma} T^p_{qk}\left(\underset{\rho}{\mathbf{b}}, L^*\right) = 0. \tag{16}$$

When we substitute for T from (13), every term in the resulting expression contains a product of the form

$$B^i_{st} = \underset{\sigma}{\bar{b}^i_p}\, \underset{\rho}{b^p_{r;s}}\, \underset{\rho}{b^r_t} \quad (\rho \neq \sigma);$$

I shall show that every such product vanishes. As a consequence of the identities $\mathbf{b}^2 = \mathbf{b}$, $\mathbf{b}+\bar{\mathbf{b}} = \mathbf{I}$ for each $\mathbf{b}$ and also $\underset{\sigma}{\bar{\mathbf{b}}}\underset{\rho}{\bar{\mathbf{b}}} = \bar{\mathbf{a}}$ $(\rho \neq \sigma)$ which follows from (14), we have

$$\underset{\rho}{b}{}^{p}_{r;s}\underset{\rho}{b}{}^{r}_{t} = \underset{\rho}{\bar{b}}{}^{p}_{r}\underset{\rho}{b}{}^{r}_{t;s}$$

and therefore $$B^i_{st} = \underset{\sigma}{\bar{b}}{}^i_p\underset{\rho}{\bar{b}}{}^p_r\underset{\rho}{b}{}^r_{t;s} = \bar{a}^i_r\underset{\rho}{b}{}^r_{t;s} = a^i_{r;s}\underset{\rho}{b}{}^r_t$$

since $\bar{\mathbf{a}}\underset{\rho}{\mathbf{b}} = 0$ and $\mathbf{a}+\bar{\mathbf{a}} = \mathbf{I}$. Now D is parallel with respect to L^*, and therefore $a^i_{r;s}a^r_p = 0$ from (5′). Multiplying by $\underset{\rho}{b}{}^p_t$ and noting that $\mathbf{a}\underset{\rho}{\mathbf{b}} = \underset{\rho}{\mathbf{b}}$, we have $B^i_{st} = 0$. Thus every term on the left of (16) vanishes, which concludes the proof that D^σ $(\sigma = 1,\ldots, m)$ is parallel with respect to L.

In proving this result we have not used any integrability conditions, so that L makes $D_1,\ldots, D_m$ parallel whether or not they are integrable. If the system of distributions is integrable, then L is symmetric, but not otherwise.

The connexion given by (15) for fixed Γ is not the only one having the required properties, but as in § 4 the most general is given by (15) with Γ as an arbitrary symmetric connexion over M.

6. In the construction of the connexion L in the last section, the most natural 'background' connexion to refer to was L^*. It is, however, a simple matter to express L directly in terms of the symmetric connexion Γ chosen arbitrarily in the first place. We have $L^* = \Gamma+T(\mathbf{a}, \Gamma)$, and it can easily be verified from an identity of the form (6′) that

$$\underset{\rho}{b}{}^i_{j;k} = a^i_r\underset{\rho}{b}{}^r_{j,k}.$$

Hence $$T^i_{jk}\big(\underset{\rho}{\mathbf{b}}, L^*\big) = a^i_r T^r_{jk}\big(\underset{\rho}{\mathbf{b}}, \Gamma\big),$$

and (15) becomes

$$L^i_{jk} = \Gamma^i_{jk}+T^i_{jk}(\mathbf{a}, \Gamma)+a^i_r\sum_{\rho=1}^{m}T^r_{jk}\big(\underset{\rho}{\mathbf{b}}, \Gamma\big). \tag{17}$$

It is again seen at once that L is symmetric when D, D^ρ $(\rho = 1,\ldots, m)$ are integrable because then $T(\mathbf{a}, \Gamma)$ and $T\big(\underset{\rho}{\mathbf{b}}, \Gamma\big)$ are symmetric by the result of § 4. It can also be verified directly from (17) that $D_1,\ldots, D_m$ are parallel with respect to L.

7. If the system of distributions $\{D_\sigma\}$ of § 5 is complete, then $\mathbf{a} = \mathbf{I}$, $\bar{\mathbf{a}} = \mathbf{0}$, and $\underset{\sigma}{\mathbf{b}} = \underset{\sigma}{\bar{\mathbf{a}}}$. In this case $L^* = \Gamma$, and from (15) the connexion

$$\Gamma + \sum_{\rho=1}^{m} T\left(\underset{\rho}{\bar{\mathbf{a}}}, \Gamma\right)$$

has the property that it makes every D_σ parallel and is symmetric if the system $\{D_\sigma\}$ is integrable.

In particular, if $m = 2$, so that we have a disjoint complementary pair D, $\bar{D}$ as in § 4, with associated tensors $\mathbf{a}$, $\bar{\mathbf{a}}$, the connexion

$$\Gamma + T(\mathbf{a}, \Gamma) + T(\bar{\mathbf{a}}, \Gamma)$$

makes both D and $\bar{D}$ parallel and is symmetric if D and $\bar{D}$ are both integrable.

VALUED LINEAR SPACES

By K. A. H. GRAVETT (*Oxford*)

[Received 16 August 1955]

1. Introduction

THE main result of this paper is an embedding theorem (Theorem 3) for valued linear spaces (defined below). Hahn's embedding and completeness theorems for ordered abelian groups are consequences of this theorem, and I hope to publish details of their derivations at a later date. My method of proof of Theorem 3 is a by-product of a study of generalizations of the structure-theory of valuations.† The aspect of the structure-theory of valuations here involved is that due to Kaplansky (**2**) and most of the results of this paper are analogues to results of Kaplansky's. Theorem 3—with W for M (see below)—is a special case of the main theorem of Conrad (**1**) although my method of proof is radically different from Conrad's. I believe that Conrad's results can be derived from a generalization of the methods of this paper.

For an account, together with full references, of the classical theory of valuations the reader is referred to the book by Schilling (**5**).

2. Preliminaries

If X and Y are sets, then $X \setminus Y$ denotes the set of all elements of X that do not belong to Y.

Let Δ be a set. A binary relation '$\leqslant$' of the set Δ is said to be an *order* of Δ (and Δ, more precisely $\{\Delta, \leqslant\}$, is said to be an *ordered set*) if, for each $x, y, z \in \Delta$,

(i) $x \leqslant y$ or $y \leqslant x$ (and hence $x \leqslant x$);
(ii) $x \leqslant y$ and $y \leqslant x$ imply $x = y$;
(iii) $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$.

A subset Σ of an ordered set Δ is said to be *dually-wellordered* if each non-empty subset Σ' of Σ has a maximum element.

All linear spaces considered are over one and the same field K. The zero element of an arbitrary linear space is denoted by 0.

† K. A. H. Gravett (Ph.D. thesis 1954, University of London) unpublished. I wish to express my gratitude to Professor R. Rado, who supervised my research for this thesis, for his advice and encouragement.

3. Definitions

Our definition of a valuation of a linear space is based on Krull's concept of a valuation of a field (**3**). However, a valued field, considered as a linear space over a subfield, is not in general a valued linear space.

Let L be a linear space and let Δ be an ordered set with minimum element μ. A mapping d of L onto Δ is said to be a *valuation* of L (and L, more precisely $\{L, \Delta, d\}$, is said to be a *valued* linear space) if, for each $x, y \in L$,

(i) $d(x) = \mu$ if and only if $x = 0$;
(ii) $d(x) = d(kx)$ for each non-zero element k of K;
(iii) $d(x+y) \leqslant \max(d(x), d(y))$.

It follows that, if $d(x) < d(y)$, then $d(x+y) = d(y)$.

Let $\{L_1, \Delta_1, d_1\}$ and $\{L_2, \Delta_2, d_2\}$ be valued linear spaces. An isomorphism f of the linear space L_1 into the linear space L_2 is said to be a *valuation-isomorphism* of the valued linear space L_1 into the valued linear space L_2 if Δ_1 is an ordered subset of Δ_2 and, for each $x \in L_1$, $d_1(x) = d_2(f(x))$.

Let $\{L, \Delta, d\}$ be a valued linear space. For each $\delta \in \Delta \setminus \{\mu\}$, denote by $A(\delta)$ the set of all elements x of L for which $d(x) \leqslant \delta$ and by $B(\delta)$ the set of all elements x of L for which $d(x) < \delta$. Further, put

$$A(\mu) = B(\mu) = \{0\}.$$

For each $\delta \in \Delta$, $A(\delta)$ and $B(\delta)$ are linear subspaces of L, and $B(\delta)$ is a linear subspace of $A(\delta)$. Denote $A(\delta)/B(\delta)$, the difference linear space of $A(\delta)$ modulo $B(\delta)$, by $C(\delta)$.

4. Extensions

The important concept of this section, that of immediacy of extensions, is again based on a concept of Krull's (**3**).

Let $\{L_1, \Delta_1, d_1\}$ and $\{L_2, \Delta_2, d_2\}$ be valued linear spaces. L_2 is said to be an *extension* of L_1 if L_1 is a linear subspace of L_2, Δ_1 is an ordered subset of Δ_2 and, for each $x \in L_1$, $d_1(x) = d_2(x)$.† In this case there is, for each $\delta \in \Delta_1$, a natural identification of the linear space $C_1(\delta)$ with a linear subspace of the linear space $C_2(\delta)$ since $B_2(\delta) \cap A_1(\delta) = B_1(\delta)$.‡ The extension L_2 of L_1 is said to be an *immediate* extension of L_1 if $\Delta_1 = \Delta_2$ and, for each $\delta \in \Delta_1 = \Delta_2$, $C_1(\delta) = C_2(\delta)$.

† L_2 is said to be a *proper extension* of L_1 if, further, L_2 properly contains L_1.

‡ $A_i(\delta)$, $B_i(\delta)$, etc., for L_i are defined analogously to $A(\delta)$, $B(\delta)$, etc., respectively for L.

It is easy to show that

(i) if L_3 is an immediate extension of L_2 and L_2 is an immediate extension of L_1, then L_3 is an immediate extension of L_1;

(ii) if L_3 is an extension of L_2, L_2 is an extension of L_1, and L_3 is an immediate extension of L_1 then L_3 is an immediate extension of L_2;

(iii) if L_2 is a proper extension of L_1, then L_2 is an immediate extension of L_1 if and only if to each $x \in L_1$ and $y \in L_2 \setminus L_1$ there corresponds $z \in L_1$ for which $d(x-y) > d(z-y)$.

I shall not employ the following result and therefore omit its proof: *to each valued linear space L there corresponds one and, within valuation-isomorphism over L, only one valued linear space which is a maximal immediate extension of L.*

5. Pseudo-convergence

The concepts of 'pseudo-convergence' for valued fields were first formulated by Ostrowski (**4**) and then extended and used to great effect by Kaplansky (**2**). Theorem 1 of this section is one of our main tools in the study of valued linear spaces.

Let $\{L, \Delta, d\}$ be a valued linear space. A subset $S = \{x_r : r \in R\}$ of L, indexed by an arbitrary dually-wellordered set R without a minimum element, is said to be *pseudo-convergent* if, for each $r, s, t \in R$ with $r > s > t$, $d(x_r - x_s) > d(x_s - x_t)$. It follows that

$$d(x_r - x_s) = d(x_r - x_t), \ = \delta_r \text{ say.}$$

An element x of L is said to be a *pseudo-limit* of S if, for each $r \in R$, $d(x_r - x) = \delta_r$. L is said to be *pseudo-complete* if each pseudo-convergent set of L has at least one pseudo-limit in L.

The proofs of Lemmas 1 and 2 closely follow the proofs of the analogous results of the classical theory [see (**2**)] and are therefore omitted.

LEMMA 1. *Let L be a valued linear space and let $S = \{x_r : r \in R\}$ be a pseudo-convergent set of L. If 0 is a pseudo-limit of S, then, for each r, $s \in R$ with $r > s$, $d(x_r) > d(x_s)$. If 0 is not a pseudo-limit of S, then there exists $r_0 \in R$ such that, for each $r \in R$ with $r_0 \geqslant r$, $d(x_r) = d(x_{r_0}) = \text{ult}\, S$ say, and, if x is a pseudo-limit of S, then $d(x) = \text{ult}\, S$.*

LEMMA 2. *Let L_1 and L_2 be valued linear spaces and let L_2 be an immediate proper extension of L_1. Then to each $y \in L_2 \setminus L_1$ there corresponds a pseudo-convergent set $S = \{x_r : r \in R\}$ of L_1 without a pseudo-limit in L_1 but with pseudo-limit y in L_2.*

From Lemma 2 it follows that, if a valued linear space L is pseudo-complete, then L has no proper immediate extension. The converse is true, but we shall not require it.

THEOREM 1. *Suppose that*

(i) *L_i and L_i' are valued linear spaces and L_i' is an immediate extension of L_i for $i = 1, 2$;*
(ii) *f is a valuation-isomorphism of L_1 onto L_2;*
(iii) *L_2' is pseudo-complete.*

Then there is a valuation-isomorphism f' of L_1' into L_2' continuing the valuation-isomorphism f of L_1 onto L_2. Such a valuation-isomorphism f' is a valuation-isomorphism of L_1' onto L_2' if and only if L_1' is pseudo-complete.

Proof. Zorn's lemma together with the following observations yield a proof of Theorem 1.

Suppose that

(i) M_i is a linear subspace of L_i' containing L_i for $i = 1, 2$;
(ii) L_1' is a proper extension of M_1;
(iii) g is a valuation-isomorphism of M_1 onto M_2 continuing f.

Let y_1 be an arbitrary, but fixed, element of $L_1' \setminus M_1$. L_1' is an immediate extension of M_1 and hence, by Lemma 2, there exists a pseudo-convergent set $S = \{x_r : r \in R\}$ of M_1 without a pseudo-limit in M_1 but with pseudo-limit y_1 in L_1'. Since g is a valuation-isomorphism, $g(S) = \{g(x_r) : r \in R\}$ is a pseudo-convergent set of M_2 without a pseudo-limit in M_2 but with a pseudo-limit, y_2 say, in L_2' (for L_2' is pseudo-complete). Denote by M_i' the linear subspace of L_i' generated by M_i and y_i for $i = 1, 2$ and denote by g' the (unique) isomorphism of the linear space M_1' onto the linear space M_2' continuing g and such that $g'(y_1) = y_2$. I show that g' is a valuation-isomorphism of M_1' onto M_2'. Let $y = x + ky_1$, where $x \in M_1$ and $k \in K \setminus \{0\}$, be an (arbitrary) element of $M_1' \setminus M_1$. $S(y) = \{x + kx_r : r \in R\}$ is a pseudo-convergent set of M_4 and 0 is not a pseudo-limit of $S(y)$ (for otherwise $-k^{-1}x \in M_1$ would be a pseudo-limit of S). It follows that $d(y) = \operatorname{ult} S(y)$ and, similarly, $d(g'(y)) = \operatorname{ult} S(g'(y))$, where $S(g'(y))$ is the pseudo-convergent set $\{g'(x) + kg'(x_r) : r \in R\}$ of M_2. However, since g' is a valuation-isomorphism of M_1 onto M_2, $\operatorname{ult} S(y) = \operatorname{ult} S(g'(y))$ and hence $d(y) = d(g'(y))$. Thus g' is a valuation-isomorphism of M_1' onto M_2'. This completes the observations.

6. Direct sums of ordered systems of linear spaces

In this section I give some examples of valued linear spaces that are important for the following section.

Let $\mathscr{L} = \{L_\delta : \delta \in \Delta\}$ be a set of linear spaces indexed by an ordered set Δ with minimum element μ for which $L_\mu = \{0\}$. I refer to $\mathscr{L}$ as an *ordered system* of linear spaces.

Denote by D, more precisely $D(\mathscr{L})$, the set of all mappings x of Δ into $\bigcup_{\delta \in \Delta} L_\delta$ for which $x(\delta) \in L_\delta$ for all $\delta \in \Delta$. For each $x, y \in D$, $k \in K$, and $\delta \in \Delta$, put $(x+y)(\delta) = x(\delta)+y(\delta)$ and $(kx)(\delta) = kx(\delta)$. Evidently D is a linear space.

For each $x \in D$, denote by Nx the set of all elements δ of Δ for which $x(\delta) \neq 0$. If Nx has a maximum element, put $d(x) = \max Nx$. If Nx is empty (i.e. $x = 0$), put $d(x) = \mu$. A subset E of D is said to be a 'b-subset' (of D) if $d(x-y)$ exists for all $x, y \in E$. By an obvious extension of the definitions the concepts of pseudo-convergence may be applied to b-subsets. A linear subspace L of D is said to be a 'b-linear subspace' (of D) if L is a b-subset. Clearly d is a valuation of each b-linear subspace into Δ. By Zorn's lemma, each b-subset is contained in at least one maximal b-subset and each b-linear subspace is contained in at least one maximal b-linear subspace.

Denote by W, more precisely $W(\mathscr{L})$, the set of all elements x of D for which Nx is a dually-wellordered subset of Δ. Denote by F, more precisely $F(\mathscr{L})$, the set of all elements x of D for which Nx is a finite subset of Δ. W and F are b-linear subspaces and F is a linear subspace of W. W and F are said to be the *wellordered-sum* and the *finite-sum* respectively of the ordered system of linear spaces $\mathscr{L}$. It is not difficult to show that (i) if L is a b-linear subspace containing F, then L is an immediate extension of F, (ii) W is a maximal b-subset and, *a fortiori*, a maximal b-linear subspace.

LEMMA 3. *If M is a maximal b-linear subspace, then M is a maximal b-subset.*

Proof. Let y be an element of $D \setminus M$. There correspond $x, x' \in M$ and $k \in K \setminus \{0\}$ such that $d((x+ky)-x')$, and hence $d(y-k^{-1}(x'-x))$, do not exist. It follows that $M \cup \{y\}$ is not a b-subset and hence that L is a maximal b-subset.

LEMMA 4. *A maximal b-subset E is pseudo-complete.*

COROLLARY. *A maximal b-linear subspace is pseudo-complete.*

Proof. Let $S = \{x_r : r \in R\}$ be a pseudo-convergent set of E. Let δ be an element of Δ. If there exist $r, s \in R$, with $r > s$, such that $\delta > d(x_r - x_s)$ then, for all $t \in R$ with $r > t$,

$$x_r(\delta) = x_t(\delta) = x(\delta), \text{ say.}$$

Otherwise put $x(\delta) = 0$. If $x \in E$, then x is a pseudo-limit of S. If $x \notin E$, then there corresponds $y \in E$ such that $d(x-y)$ does not exist, and then y is a pseudo-limit of S.

THEOREM 2. *Let M_1 and M_2 be maximal b-linear subspaces containing F. Then there is a valuation-isomorphism of M_1 onto M_2 over F.*

Proof. M_i is an immediate extension of F and is pseudo-complete for $i = 1, 2$. Hence by Theorem 1 there is a valuation-isomorphism of M_1 onto M_2 over F.

7. An embedding-theorem for valued linear spaces

I prove in Theorem 3 that a valued linear space L is valuation-isomorphic to a b-linear subspace of the direct sum of the ordered system of linear spaces $\{C(\delta) : \delta \in \Delta\}$. Lemma 5 is preliminary to Theorem 3.

LEMMA 5. *Let $\{L, \Delta, d\}$ be a valued linear space. Then there is a valuation-isomorphism ϕ of the finite-sum F of the ordered system of linear spaces $\{C(\delta) : \delta \in \Delta\}$ into the valued linear space L for which L is an immediate extension of the linear subspace $\phi(F)$ of L.*

Proof. For each $\delta \in \Delta$, there is a natural isomorphism f_δ of $C(\delta)$ onto the linear subspace of F consisting of all elements x of F for which either Nx is empty or $Nx = \{\delta\}$; f_δ is defined by putting, for each $c \in C(\delta)$, $f_\delta(c) = x$ ($\in F$) where $x(\delta) = c$ and $x(\delta') = 0$ for all $\delta' \in \Delta \setminus \{\delta\}$. Corresponding to each element $\delta \in \Delta$, select a linear subspace $C^0(\delta)$ of L such that $C^0(\delta) \subseteq A(\delta)$ and each coset X of $A(\delta)$ modulo $B(\delta)$ contains one and only one element, $g_\delta(X)$ say, of $C^0(\delta)$. Evidently g_δ is an isomorphism of $C(\delta)$ onto $C^0(\delta)$. We now define a mapping ϕ of F into L. Put $\phi(0) = 0$. Let x be an element of $F \setminus \{0\}$. Then, uniquely,

$$x = \sum_{i=1}^{n} f_{\delta_i}(c_i),$$

where

$$\delta_1, \delta_2, \ldots, \delta_n \in \Delta,\ \delta_1 > \delta_2 > \ldots > \delta_n,\ c_i \in C(\delta_i) \setminus \{0\}\ (i = 1, 2, \ldots, n).$$

Put $\phi(x) = \sum_{i=1}^{n} g_{\delta_i}(c_i)$. It is not difficult to verify that the mapping ϕ has the required properties.

THEOREM 3. *Let $\{L, \Delta, d\}$ be a valued linear space and denote by ϕ the valuation-isomorphism of Lemma 5 of $F(\{C(\delta): \delta \in \Delta\})$ into L. Let M be a maximal b-linear subspace of $D(\{C(\delta): \delta \in \Delta\})$ containing F. Then there exists a valuation-isomorphism ψ of L into M continuing the valuation-isomorphism ϕ^{-1} of $\phi(F)$ onto F.*

Proof. L is an immediate extension of $\phi(F)$ and M is an immediate extension of F and is pseudo-complete. An application of Theorem 1 completes the proof.

The following 'construction' yields an alternative proof of Theorem 3 by 'exhibiting' a valuation-isomorphism h of L into a b-linear subspace of D containing F. The theorem then follows from previous results.

Let δ be an element of Δ. Since $C(\delta) = A(\delta)/B(\delta)$ is a linear subspace of $L/B(\delta)$ and by virtue of the construction of $C^0(\delta')$ (see Lemma 5) for each $\delta' \in \Delta$, there is a homomorphism p_δ of $L/B(\delta)$ onto $C(\delta)$ which is the identity homomorphism on $C(\delta)$ and is such that $p_\delta(x) = 0$ for each coset x of L modulo $B(\delta)$ which contains an element of $C^0(\delta')$ for every $\delta' \in \Delta$ with $\delta' > \delta$. Denote by q_δ the homomorphism of L onto $C(\delta)$ which is the natural homomorphism of L onto $L/B(\delta)$ followed by the homomorphism p_δ of $L/B(\delta)$ onto $C(\delta)$. We now define the valuation-isomorphism h. Let x be an element of L. There exists, uniquely, an element $h(x)$ of D such that, for each $\delta \in \Delta$, $(h(x))(\delta) = q_\delta(x)$. It is not difficult to verify that $h(L)$ is a b-linear subspace of D containing F and that h is a valuation-isomorphism of L onto $h(L)$.

[*Added* 16 *January* 1956.] With regard to Theorem 2 it is not difficult to prove that *every* maximal b-linear subspace M of W is an immediate extension of a valuation-isomorph of F. It follows that *any* two maximal b-linear subspaces are valuation-isomorphic.

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INTEGRABILITY THEOREMS FOR POWER SERIES

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1. Suppose that c_n is real for all n and that the series

$$f(x) = \sum_0^\infty c_n x^n \quad (0 \leqslant x < 1) \tag{1.1}$$

is convergent in the range indicated. Heywood [(**2**), Theorem 3 (i)] has proved that, *if* $c_n \geqslant 0$ *for all sufficiently large* n, *and if*

$$\sum_0^\infty c_n = 0, \tag{1.2}$$

then
$$(1-x)^{-1}f(x) \in L(0,1) \tag{1.3}$$

if and only if
$$\sum c_n \log n \quad \textit{converges.} \tag{1.4}$$

The main object of this note is to obtain sufficient conditions for the equivalence of (1.3) and (1.4) when c_n is not assumed to be ultimately non-negative.

We first observe that (1.2) and (1.4), taken together, imply

$$s_n \log n \to 0 \quad (n \to \infty), \tag{1.5}$$

where
$$s_n = \sum_0^n c_r. \tag{1.6}$$

For, as soon as N is so large that

$$\left|\sum_{N+1}^{N+p} c_n \log n\right| < \epsilon$$

for all integers $p \geqslant 1$, we have by Abel's lemma [Titchmarsh (**3**), 6]

$$\left|\sum_{N+1}^{N+p} c_n\right| = \left|\sum_{N+1}^{N+p} c_n \log n (\log n)^{-1}\right| < \epsilon(\log N)^{-1}.$$

Letting $p \to \infty$ we find, by (1.2), that

$$|s_N| = \left|\sum_{N+1}^{\infty} c_n\right| \leqslant \epsilon(\log N)^{-1},$$

which gives (1.5).

This remark suggests that we should investigate the relation between (1.3) and (1.4) under the overall assumption (1.5). Actually I replace (1.5) by

$$s_n \log n \to \alpha, \tag{1.7}$$

where α is some finite limit; and moreover I find it more convenient at first to replace (1.3) by

$$I_\rho \equiv \int_0^\rho (1-x)^{-1}f(x)\,dx \to I \tag{1.8}$$

as $\rho \to 1$ from the left, where I denotes some finite limit. I then prove the following theorem.

THEOREM I. *Of the three statements* (1.4), (1.7), *and* (1.8), *any two, taken together, imply the remaining one.*

We seek a theorem which, under suitable conditions, asserts the equivalence of (1.3) and (1.4). To this end I impose on the c_n a further restriction which secures that (1.3) and (1.8) are equivalent. Using a standard notation I express this restriction in terms of the sums $S_n^{(p)}$, defined for all non-negative integers n and p by the identity

$$(1-x)^{-p-1}f(x) \equiv \sum_{n=0}^{\infty} S_n^{(p)} x^n \quad (|x| < 1) \tag{1.9}$$

(so that, in particular, $S_n^{(0)} = s_n$). Then we have

THEOREM II. *Suppose that there is an integer* $p \geqslant 0$ *such that, for all sufficiently large* n, $S_n^{(p)} \geqslant 0$. *Then, of the three statements* (1.3), (1.4), *and* (1.7), *any two, taken together, imply the remaining one.*

I state also the following Tauberian theorem, which is easily deduced from Theorem I.

THEOREM III. *Suppose that*

$$\sum_1^\infty c_n\left(\sum_1^n \frac{\rho^r}{r}\right) \to l \tag{1.10}$$

as $\rho \to 1$ *from the left, where* l *is finite and the series is convergent for* $0 \leqslant \rho < 1$, *and suppose also that*

$$\left(\sum_{n+1}^{\infty} c_r\right)\log n \to \beta \tag{1.11}$$

as $n \to \infty$, *where* β *is finite. Then*

$$\sum_1^\infty c_n\left(\sum_1^n \frac{1}{r}\right) = l. \tag{1.12}$$

It should perhaps be observed that the series in (1.12) cannot converge at all unless (1.11) is true with $\beta = 0$. This is easily proved by an argument involving Abel's lemma, similar to that which we have already used, coupled with (2.1) below.

2. In this section I prove three simple lemmas.

LEMMA 1. *The series*

$$\sum c_n \log n, \qquad \sum c_n \Bigl(\sum_1^n \frac{1}{r}\Bigr)$$

are either both convergent or both divergent.

We have

$$\sum_1^n \frac{1}{r} = \log n + \gamma + u_n, \tag{2.1}$$

where γ is Euler's constant and u_n decreases steadily to 0 as $n \to \infty$ [Copson (**1**), 214]. By Abel's test for convergence, the convergence of either of the series in Lemma 1 implies the convergence of $\sum c_n$ and $\sum c_n u_n$. Lemma 1 now follows at once from (2.1).

LEMMA 2. *If* (1.7) *holds, then the series*

$$\sum \frac{s_n}{n+1}, \qquad \sum c_n \Bigl(\sum_1^n \frac{1}{r}\Bigr)$$

are either both convergent or both divergent. If the series are both convergent, then (1.7) *holds.*

We have

$$\sum_0^N \frac{s_n}{n+1} = \frac{s_N}{N+1} + \sum_0^{N-1} \frac{1}{n+1}\Bigl(s_N - \sum_{n+1}^N c_r\Bigr)$$

$$= s_N \sum_0^N \frac{1}{n+1} - \sum_1^N c_n \Bigl(\sum_1^n \frac{1}{r}\Bigr),$$

and Lemma 2 follows at once from this and (2.1).

LEMMA 3. *The statements* (1.2) *and* (1.8) *imply*

$$\sum_0^\infty \frac{s_n}{n+1} = I.$$

If $0 < \rho < 1$, we have

$$I_\rho = \int_0^\rho \Bigl(\sum_0^\infty s_n x^n\Bigr)\, dx = \sum_0^\infty \frac{s_n}{n+1} \rho^{n+1} \tag{2.2}$$

since the series $\sum s_n x^n$ converges uniformly for $0 \leqslant x \leqslant \rho$. I now apply Tauber's theorem [Titchmarsh (**3**), 10] to the third member of (2.2). By (1.8) and (2.2), the sum of this series approaches I as $\rho \to 1$ from the left, and by (1.2), $s_n/(n+1) = o(n^{-1})$. Therefore Lemma 3 follows immediately from Tauber's theorem.

3. To prove Theorem I, first suppose that (1.4) and (1.7) hold. By (1.4) and Lemma 1, the second of the two series in Lemma 1 is convergent, and so, by (1.7) and Lemma 2, $\sum s_n/(n+1)$ is convergent. Hence by Abel's theorem on power series [Titchmarsh (**3**), 9], applied to the third member of the identity (2.2), we obtain (1.8).

Next, suppose that (1.7) and (1.8) hold. Then by Lemma 3, since (1.7) contains (1.2), it follows that $\sum s_n/(n+1)$ is convergent, and so the second of the two series in Lemma 2 is convergent. This implies (1.4), by Lemma 1.

Finally, suppose that (1.4) and (1.8) hold. Then by (1.4) (as in the proof of Lemma 1), $\sum c_n$ is convergent, and so by Abel's theorem

$$f(x) \to \sum_0^\infty c_n$$

as $x \to 1$ from the left. From this combined with (1.8), (1.2) follows easily. But it has been shown in § 1 that (1.2) and (1.4) imply (1.5), which is (1.7) with $\alpha = 0$. This completes the proof of Theorem I.

To prove Theorem II, suppose that, for a certain integer $p \geqslant 0$, $S_n^{(p)} \geqslant 0$ for all $n > N$. Then, by (1.9),

$$(1-x)^{-1}f(x) = (1-x)^p \sum_{n=0}^{N} S_n^{(p)} x^n + (1-x)^p \sum_{n=N+1}^{\infty} S_n^{(p)} x^n.$$

On the right-hand side of this identity, the first term is a polynomial and the second is non-negative for $0 \leqslant x < 1$. Hence (1.3) and (1.8) are equivalent. Theorem II follows from this remark and Theorem I.†

To prove Theorem III, suppose that (1.10) and (1.11) hold. Choose c_0 so that (1.2) is true, and define $f(x)$ and s_n by (1.1) and (1.6) respectively. Then (1.7) (with $\alpha = -\beta$) follows from (1.2) and (1.11). Further, following Heywood (**2**), we observe that by (1.2),

$$I_\rho = -\int_0^\rho \left(\sum_1^\infty c_n \frac{1-x^n}{1-x}\right) dx \quad (0 < \rho < 1).$$

The series converges uniformly for $0 \leqslant x \leqslant \rho$, and so we may integrate term by term to obtain

$$I_\rho = -\sum_1^\infty c_n \left(\sum_1^n \frac{\rho^r}{r}\right).$$

† The idea used in this proof also yields a theorem of a different type, as an immediate corollary of Theorem 1 of Heywood (**2**): *If there is an integer $p \geqslant 0$ such that, for all sufficiently large n, $S_n^{(p)} \geqslant 0$, then* (1.3) *holds if and only if $\sum_n n^{-p-1} S_n^{(p)}$ is convergent.*

Therefore, by (1.10), (1.8) is true. Now (1.7), (1.8), and Theorem I give (1.4), and so by Lemma 1 the series in (1.12) is convergent.

To see that the sum of this series is l, we write

$$\sum_1^\infty c_n\left(\sum_1^n \frac{\rho^r}{r}\right) = \sum_1^\infty a_n v_n(\rho), \tag{3.1}$$

where $$a_n = c_n \sum_1^n \frac{1}{r}, \qquad v_n(\rho) = \sum_1^n \frac{\rho^r}{r} \Big/ \sum_1^n \frac{1}{r}.$$

Then (i) $\sum a_n$ is convergent, and (ii), for $0 \leqslant \rho \leqslant 1$, $v_n(\rho)$ does not increase when n increases, and $0 \leqslant v_1(\rho) \leqslant 1$. It is known [Copson (1), 99] that (i) and (ii) imply the uniform convergence of $\sum a_n v_n(\rho)$ for $0 \leqslant \rho \leqslant 1$. Since, for each n, $v_n(\rho)$ is continuous for $0 \leqslant \rho \leqslant 1$, the sum of the series (3.1) is continuous in the same range, and (1.12) follows from this. This proves Theorem III.

My thanks are due to the referee for some improvements in the presentation of this note.

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